Fundamental invariants of improper symplectic reflection groups

RPTU Rheinland-Pfälzische Technische Universität Kaiserslautern Landau | DFG Deutsche Forschungsgemeinschaft | TRR 195 SYMBOLIC TOOLS

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Notation

- $\mathfrak{h}$ a $\mathbb{C}$-vector space of dimension n
- $\mathfrak{h}^*$ the dual, that is, the linear maps $\mathfrak{h} \to \mathbb{C}$
- $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*]$ the symmetric algebra of $\mathfrak{h}^* \oplus \mathfrak{h}$ $\Rightarrow$ after choosing a basis for $\mathfrak{h}$: $\mathbb{C}[\mathfrak{h} \oplus \mathfrak{h}^*] = \mathbb{C}[x_1 \dots x_n, y_1 \dots y_n] = \mathbb{C}[x, y]$
- $\mathcal{W} \subseteq \mathrm{GL}(\mathfrak{h})$ a complex reflection group

Diagonal Invariants

$\mathcal{W}$ has a *diagonal action* on $\mathbb{C}[x, y]$ by

$$(A, f(x, y)) \mapsto A \star f(x, y) := f(A\,x, (A^t)^{-1}\,y)$$

- $\mathbb{C}[x, y]^{\mathcal{W}}$ the polynomial invariants
- $\mathbb{C}(x, y)^{\mathcal{W}}$ the rational invariants, that is, the field of fractions of $\mathbb{C}[x, y]^{\mathcal{W}}$

Problem: As a $\mathbb{C}$-algebra, $\mathbb{C}[x, y]^{\mathcal{W}}$ is finitely generated. $\Rightarrow$ How to compute the generators?

Motivation

- Classification of reflection groups:
  $\mathcal{W}$ is a complex reflection group over $\mathfrak{h}$.
  $\Leftrightarrow$ $\mathbb{C}[x]^{\mathcal{W}}$ is a polynomial ring.
  $\Leftrightarrow$ $\mathfrak{h}/\mathcal{W} = \mathrm{Spec}(\mathbb{C}[x]^{\mathcal{W}})$ is smooth.
  However, $\mathcal{W}$ represented over $\mathfrak{h} \oplus \mathfrak{h}^*$ as

  $$\mathcal{W} \ni A \mapsto \begin{pmatrix} A & 0 \\ 0 & (A^t)^{-1} \end{pmatrix}$$

  is NOT generated by reflections.
  Instead, we have a symplectic form

  $$((x, y), (x', y')) \mapsto y'(x) - y(x').$$

  on $\mathfrak{h} \oplus \mathfrak{h}^*$ and $(\mathfrak{h} \oplus \mathfrak{h}^*)/\mathcal{W}$ is a singular symplectic variety, see [1, 2].
- Hamiltonian equations of motion:
  A Lie bracket on $\mathbb{C}[x, y]$ is defined by

  $$(f, g) \mapsto \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i},$$

  turning $\mathbb{C}[x, y]$ into the *Poisson algebra*.

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Acknowledgements

This work is supported by the DFG research centre SFB-TRR 195: Symb. Tools in Math. and their Application.

King's Algorithm

... computes the generators ("fundamental invariants") for $\mathbb{C}[V]^{\mathcal{W}}$, where V is a finite dimensional $\mathcal{W}$-module, for example $V = \mathfrak{h} \oplus \mathfrak{h}^*$.

Set $F := \emptyset$, $G := \emptyset$, $\preceq$ the deg rev lex ordering on $\mathbb{C}[V]$. For $1 \le d \le |\mathcal{W}|$ do:

1. $G := G \cup \{\mathrm{NF}(h) \,|\, h = \mathrm{spoly}(f, g),$ $f, g \in G,\ \deg(h) = d\}$.
2. $M := \{x^\alpha \in \mathbb{C}[V] \,|\, |\alpha| = d,$ $\forall g \in G : \mathrm{LM}(g)$ *does NOT divide* $x^\alpha\}$.
3. *If $M = \emptyset$, stop.*
4. *For $t \in M$, set $f := \frac{1}{|\mathcal{W}|}\sum_{A \in \mathcal{W}} A \star t$.*
5. *If $\mathrm{NF}(f) \neq 0$, add f to F, $\mathrm{NF}(f)$ to G.*

F is a set of fundamental invariants for $\mathbb{C}[V]^{\mathcal{W}}$.

Degree Principles

A polynomial $f \in \mathbb{C}[x, y]$ has *bidegree* $\mathrm{Deg}(f) := (\deg_x(f), \deg_y(f)) \in \mathbb{N}^2$.

For $f(x, y) = \sum_\alpha c_\alpha\, p_\alpha(x)\, q_\alpha(y)$, set

$$\Psi(f)(x, y) := f(y, x) = \sum_\alpha c_\alpha\, p_\alpha(y)\, q_\alpha(x).$$

Theorem 1: The involution Ψ takes invariants to invariants: For $f \in \mathbb{C}[x, y]^{\mathcal{W}}$ with $\mathrm{Deg}(f) = (d, e)$, we have $\Psi(f) \in \mathbb{C}[x, y]^{\mathcal{W}}$ and $\mathrm{Deg}(\Psi(f)) = (e, d)$.

Theorem 2: There is a system F of fundamental invariants for $\mathbb{C}[x, y]^{\mathcal{W}}$ with $\Psi(F) = F$.

Improving on King's

- $d \le |\mathcal{W}|$ is suboptimal: It suffices to take

  $$d \le \inf\left\{ k \in \mathbb{N} \;\middle|\; \mathbb{C}[x, y] = \langle \bigoplus_{\ell=0}^{k} \mathbb{C}[x, y]_\ell \rangle \right\}.$$

  Compute this bound efficiently.
- Exploit known results on the coinvariants $\mathbb{C}[x, y]/I$, where I is the ideal generated by the invariants without constant term [3, 4].
- Compute diagonal polynomial invariants from rational invariants [5].
- Exploit the symmetry given by the Degree Principles: King's Algorithm does NOT do that (see Example).
- King's algorithm is based on Gröbner bases. Symmetry adapted bases and H-bases [6, 7, 8] on the other hand preserve symmetry.

Relative Invariants

Given a character $\chi : \mathcal{W} \to \mathbb{C} \setminus \{0\}$, we call $f \in \mathbb{C}[x, y]$ a *relative invariant*, if, for all $A \in \mathcal{W}$, we have $A \star f = \chi(A)\, f$. We denote this by $f \leftrightarrow \chi$.

Theorem 3: For $f \leftrightarrow \chi$, there exists $f^* \in \mathbb{C}[x, y]$ with $f^* \leftrightarrow \chi^{-1}$ and $\mathrm{Deg}(f) = \mathrm{Deg}(f^*)$.

For $\chi = 1$, this is Theorem 1.

Example

Let $a \in \mathbb{C}$ with $a^2 + a + 1 = 0$, that is, $a^3 = 1$. We consider the complex reflection group $\mathcal{W} := G_4$ (in the Shephard-Todd-classification), which is the group generated over $\mathfrak{h} := \mathbb{C}^2$ by the matrices

$$A_1 = \begin{pmatrix} a & 0 \\ -a-1 & 1 \end{pmatrix} \quad \text{and} \quad A_2 = \begin{pmatrix} 1 & a+1 \\ 0 & a \end{pmatrix}.$$

The group $\mathcal{W}$ has order 24 and any minimal set of fundamental invariants for $\mathbb{C}[x]^{\mathcal{W}}$ consists of 2 algebraically independent generators.
However, this is not true for $\mathbb{C}[x, y]^{\mathcal{W}}$: With King's algorithm, one obtains 8 fundamental invariants

$$f_1 = x_1\, y_1 + x_2\, y_2$$

$$f_2 = y_1\, y_2\, (y_1^2 - y_2^2 + (2\,a + 1)\, y_1\, y_2)$$
$$f_3 = (x_1^2 - x_2^2)\, (x_1^2 - x_2^2 + 4/3\, (2\,a + 1)\, x_1\, x_2)$$

$$f_4 = x_1\, y_1^3 + x_2\, y_2^3 - 3\, (x_2\, y_1 + x_1\, y_2) + (2\,a + 1)\, y_1\, y_2\, (x_1\, y_1 - x_2\, y_2) - (2\,a + 1)\, (x_1\, y_2^3 - x_2\, y_1^3)$$
$$f_5 = x_1^3\, y_1 + x_2^3\, y_2 - 3\, (x_2\, y_1 + x_1\, y_2) + (4\,a + 2)\, x_1\, x_2\, (x_1\, y_1 - x_2\, y_2)$$

$$f_6 = x_1^6 + x_2^6 + (4\,a + 2)\, x_1\, x_2\, (x_1^4 - x_2^4) - 5\, x_1^2\, x_2^2\, (x_1^2 + x_2^2)$$
$$f_7 = y_1^6 + y_2^6 + (4\,a + 2)\, y_1\, y_2\, (y_1^4 - y_2^4) - 5\, y_1^2\, y_2^2\, (y_1^2 + y_2^2)$$

$$f_8 = x_1^3\, y_2^3 - x_2^3\, y_1^3 - (2\,a + 1)\, (x_1\, y_1 - x_2\, y_2)\, (x_1^2\, y_2^2 - x_2^2\, y_1^2) - (x_1\, y_2 - x_2\, y_1)\, (x_1^2\, y_1^2 + x_2^2\, y_2^2 - 3\, x_1\, x_2\, y_1\, y_2),$$

forming a system F of fundamental invariants and ordered by their bidegrees

$$\mathrm{Deg}(F) = \{(1, 1), (0, 4), (4, 0), (1, 3), (3, 1), (0, 6), (6, 0), (3, 3)\}.$$

We observe that $\mathrm{Deg}(F) \subseteq \mathbb{N}^2$ is $\mathfrak{S}_2$-symmetric. We have

$$\Psi(f_1) = f_1, \Psi(f_6) = f_7, \Psi(f_8) = -f_8.$$