Symmetry Adapted Bases for Trigonometric Optimization

Tobias Metzlaff
tobias.metzlaff@rptu.de

RPTU Rheinland-Pfälzische Technische Universität Kaiserslautern Landau
DFG Deutsche Forschungsgemeinschaft
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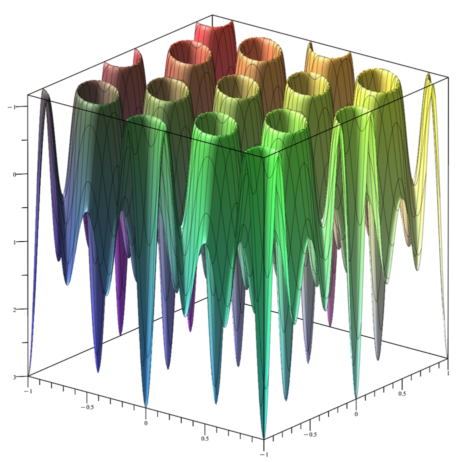
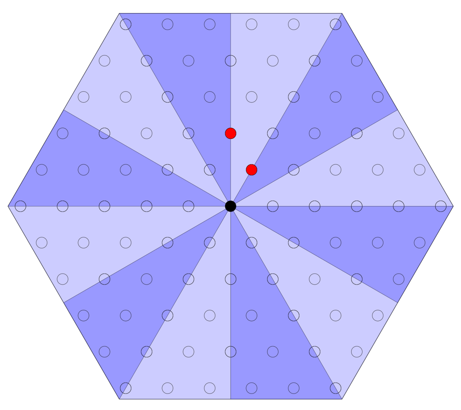
The Problem

Let $\Omega = \mathbb{Z}\omega_1 \oplus \ldots \oplus \mathbb{Z}\omega_n \subseteq \mathbb{R}^n$ be a **lattice** and $\mathcal{W} \subseteq \mathrm{O}(\mathbb{R}^n)$ be a **finite reflection group**, such that $\mathcal{W}\,\Omega = \Omega$. The goal is to compute

$$f^* := \min_{u\in\mathbb{R}^n} f(u) := \min_{u\in\mathbb{R}^n} \sum_{\mu\in\Omega} c_\mu \underbrace{\exp(-2\pi \mathrm{i}\,\langle \mu, u\rangle)}_{=:\mathfrak{e}^\mu(u)}$$

where the coefficients $c_\mu \in \mathbb{C}$ satisfy $c_{-\mu} = \overline{c_\mu}$, $c_{A\mu} = c_\mu$ whenever $A \in \mathcal{W}$ and only finitely many are nonzero.



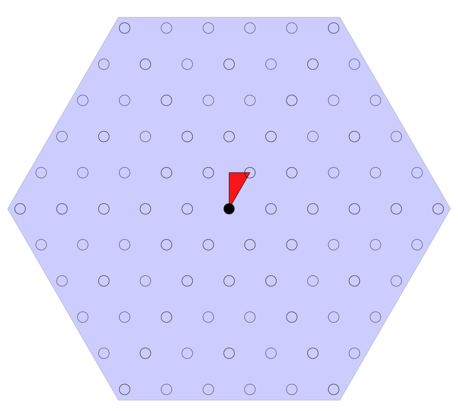


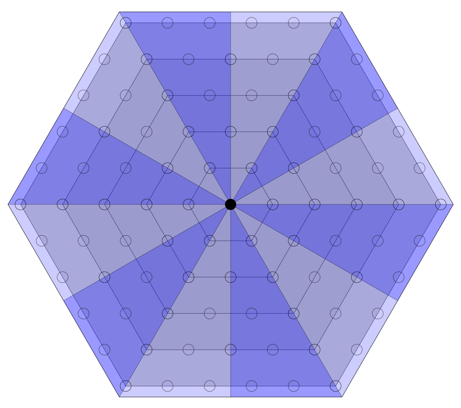
Periodicity and Degree

Denote by $\Lambda := \{\lambda \in \mathbb{R}^n \,|\, \forall\, \mu \in \Omega : \langle \mu, \lambda\rangle \in \mathbb{Z}\}$ the **dual lattice**. Then the objective function is **Λ-periodic**, that is, $f(u+\lambda) = f(u)$ for $\lambda \in \Lambda$. The group product $\mathcal{W} \ltimes \Lambda$ is semi-direct and $\triangle \subseteq \mathbb{R}^n$ shall be a **fundamental domain** containing 0. Then the finite set

$$\Omega_d := \{\mathcal{W}\,\mu \,|\, \mu \in \Omega,\ \mu/d \in \triangle\}$$

is stable under $\mathcal{W}$ and we say that f has degree $d \in \mathbb{N}$, if $\mu \in \Omega_d$ whenever $c_\mu \neq 0$ and d is minimal [1].





Semi-Definite Relaxation

When f has degree $2d \in \mathbb{N}$, then there exists a Hermitian Toeplitz matrix of size $|\Omega_d| \times |\Omega_d|$, denoted by $\mathbf{mat}(f) \in \mathrm{Toep}_d$, such that

$$f(u) = \overline{\mathbf{E}_d(u)}^t\, \mathbf{mat}(f)\, \mathbf{E}_d(u), \qquad \mathbf{E}_d := \frac{(\mathfrak{e}^\mu)^t_{\mu\in\Omega_d}}{\sqrt{|\Omega_d|}}.$$

We consider the semi-definite program

$$f_d := \min_{\mathbf{X}\in\mathrm{Toep}_d} \mathrm{Tr}(\mathbf{mat}(f)\,\mathbf{X}) \text{ s.t. } \begin{cases} \mathbf{X}\succeq 0, \\ \mathrm{Tr}(\mathbf{X})=1 \end{cases}.$$

The hierarchy $f_d \leq f_{d+1} \leq$ converges to f^* [2].

References

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Reduction via Symmetry Adapted Bases

The group $\mathcal{W}$ acts on the finite set Ω_d. Therefore, we have a left group action

$$\star : \mathcal{W} \times \mathrm{Toep}_d \to \mathrm{Toep}_d,\ (A, \mathbf{X} = (\mathbf{X}_{\mu\,\nu})) \mapsto A \star \mathbf{X} := (\mathbf{X}_{A^{-1}\mu\, A^{-1}\nu}).$$

Since f is $\mathcal{W}$-invariant, we have $A \star \mathbf{mat}(f) = \mathbf{mat}(f)$, which is denoted by $\mathbf{mat}(f) \in \mathrm{Toep}_d^{\mathcal{W}}$.

Theorem 1. *We have $f_d = f_{d,\mathcal{W}}$, where*

$$f_{d,\mathcal{W}} := \min_{\mathbf{X}\in\mathrm{Toep}_d^{\mathcal{W}}} \mathrm{Tr}(\mathbf{mat}(f)\,\mathbf{X}) \quad \text{s.t.} \quad \mathbf{X}\succeq 0,\ \mathrm{Tr}(\mathbf{X}) = 1.$$

To exploit this result, note that the action of $\mathcal{W}$ on Toep_d is induced by a linear representation $\rho : \mathcal{W} \to \mathrm{O}(\mathbb{R}^{\Omega_d})$, where $(\rho(A)\,\mathbf{x})_\mu = \mathbf{x}_{A^{-1}\mu}$ whenever $\mathbf{x} \in \mathbb{R}^{\Omega_d}$: We have $A \star \mathbf{X} = \rho(A)\,\mathbf{X}\,\rho(A)^t$. The $\rho(\mathcal{W})$-module $\mathbb{R}^{\Omega_d}$ is semi-simple and has an **isotypic decomposition**

$$\mathbb{R}^{\Omega_d} = \bigoplus_{i=1}^{h} \left(\bigoplus_{j=1}^{m_i} V_{i\,j} \right),$$

where, for all $1 \leq i \leq h$, the $V_{i\,1}, \ldots, V_{i\,m_i}$ are irreducible isomorphic $\rho(\mathcal{W})$-submodules with dimension $d_i := \dim(V_{i\,j})$ and multiplicity $m_i \in \mathbb{N}$ so that $\sum_i d_i\, m_i = |\Omega_d|$. There exists an orthogonal change of basis $\mathbf{T}$ that transforms any $\mathbf{X} \in \mathrm{Toep}_d^{\mathcal{W}}$ into a block matrix

$$\mathbf{T}^t\,\mathbf{X}\,\mathbf{T} = \begin{pmatrix} \boxed{\mathbf{X}_1} & & \\ & \ddots & \\ & & \boxed{\mathbf{X}_h} \end{pmatrix} \quad \text{with} \quad \mathbf{X}_i = \begin{pmatrix} \boxed{\tilde{\mathbf{X}}_i} & & \\ & \ddots & \\ & & \boxed{\tilde{\mathbf{X}}_i} \end{pmatrix} \in \mathbb{C}^{d_i\, m_i \times d_i\, m_i},$$

where each $\mathbf{X}_i$ consists of d_i equal blocks of size $m_i \times m_i$ [3, 4].

Theorem 2. *Assume that $\mathbf{T}^t\,\mathbf{mat}(f)\,\mathbf{T}$ has blocks $\tilde{\mathbf{F}}_i \in \mathbb{C}^{m_i\times m_i}$. Then $f_d = f_{d,\mathcal{W}} = f_{d,\mathcal{W}}^{\text{block}}$, where*

$$f_{d,\mathcal{W}}^{\text{block}} := \min_{\mathbf{X}\in\mathrm{Toep}_d^{\mathcal{W}}} \sum_{i=1}^{h} d_i\,\mathrm{Tr}(\tilde{\mathbf{F}}_i\,\tilde{\mathbf{X}}_i) \quad \text{s.t.} \quad \mathrm{Tr}(\mathbf{X}) = 1,\ \mathbf{T}^t\,\mathbf{X}\,\mathbf{T} \text{ has blocks } \tilde{\mathbf{X}}_i \succeq 0.$$

Example ($n = 1$, $\Omega = \Lambda = \mathbb{Z}$, $\mathcal{W} = \{\pm 1\}$)

The first step of the hierarchy is $\Omega_1 = \{1, 0, -1\}$ and $\mathbf{E}_1 = (\mathfrak{e}^1, 1, \mathfrak{e}^{-1})^t/\sqrt{3}$. Consider $f(u) :=$

$$\overline{\mathbf{E}_1(u)}^t \underbrace{\begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix}}_{=\mathbf{mat}(f)\in\mathrm{Toep}_1^{\mathcal{W}}} \mathbf{E}_1(u) = \mathfrak{e}^2(u) - 2\,\mathfrak{e}^1(u) + 3 - 2\,\mathfrak{e}^{-1}(u) + \mathfrak{e}^{-2}(u) = 2\,\cos(4\pi u) - 4\,\cos(2\pi u) + 3$$

with degree 2 and global minimum $f^* = f(\lambda \pm 1/6) = 0$ for $\lambda \in \mathbb{Z}$. The semi-definite relaxation is

$$f_1 = \min_{b,c\in\mathbb{C}} \mathrm{Tr}\left(\begin{pmatrix} 3 & -3 & 3 \\ -3 & 3 & -3 \\ 3 & -3 & 3 \end{pmatrix} \begin{pmatrix} 1/3 & b & c \\ \bar{b} & 1/3 & b \\ \bar{c} & \bar{b} & 1/3 \end{pmatrix} \right) \quad \text{s.t.} \quad \begin{pmatrix} 1/3 & b & c \\ \bar{b} & 1/3 & b \\ \bar{c} & \bar{b} & 1/3 \end{pmatrix} \succeq 0$$

and the optimal value $f^* = f_1 = 0$ is obtained with $b = -c = 1/6$. We have

$$\begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 1 & a & b & c \\ 0 & \bar{b} & a & b \\ -1 & \bar{c} & \bar{b} & a \end{array} \quad \xrightarrow{(-1)\,\star} \quad \begin{array}{c|ccc} & 1 & 0 & -1 \\ \hline 1 & a & \bar{b} & \bar{c} \\ 0 & b & a & \bar{b} \\ -1 & c & b & a \end{array}$$

for $a \in \mathbb{R}$ and $b, c \in \mathbb{C}$. The fixed point space $\mathrm{Toep}_1^{\mathcal{W}}$ consists of those $\mathbf{X} \in \mathrm{Toep}_1$ with $a, b, c \in \mathbb{R}$. The above action is induced by the representation $\rho : \mathcal{W} \to \mathrm{O}(\mathbb{R}^3)$, given by

$$\rho(1) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},\ \rho(-1) := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \text{ and } \mathbb{R}^3 = \left(\langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \rangle_{\mathbb{R}} \oplus \langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \rangle_{\mathbb{R}} \right) \oplus \langle \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \rangle_{\mathbb{R}}$$

is an isotypic decomposition with $h = 2, m_1 = 2, d_1 = 1, m_2 = 1, d_2 = 1$. Then

$$f_{1,\mathcal{W}}^{\text{block}} = \min_{b,c\in\mathbb{R}} \mathrm{Tr}\left(\begin{pmatrix} 3 & -3\sqrt{2} \\ -3\sqrt{2} & 6 \end{pmatrix} \begin{pmatrix} 1/3 & \sqrt{2}\,b \\ \sqrt{2}\,b & 1/3 + c \end{pmatrix} \right) \quad \text{s.t.} \quad \begin{pmatrix} 1/3 & \sqrt{2}\,b \\ \sqrt{2}\,b & 1/3 + c \end{pmatrix} \succeq 0,\ 1/3 \geq c.$$

Finally, the optimal value $f^* = f_1 = f_{1,\mathcal{W}} = f_{1,\mathcal{W}}^{\text{block}} = 0$ is recovered with $b = -c = 1/6$.