Symmetry in Trigonometric Optimization

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The Effect of Symmetry Exploitation

The goal of trigonometric optimization is to find the global minimum of a function $\mathbb{R}^n \to \mathbb{R}$ such as

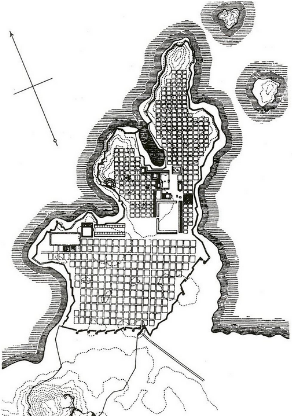
$$
\begin{aligned}
&-1 + 2/3\,(2\,\cos(2\,\pi\,x)\,\cos((-2\,x - 2\,y)\,\pi)^2\,\cos(2\,\pi\,y) + 2\,\cos(2\,\pi\,x)\,\cos(2\,\pi\,y)^2\,\cos((-2\,x - 2\,y)\,\pi)\\
&+2\,\cos(2\,\pi\,x)^2\,\cos(2\,\pi\,y)\,\cos((-2\,x - 2\,y)\,\pi) + \cos(2\,\pi\,y)^2\,\cos((-2\,x - 2\,y)\,\pi)^2 + \sin(2\,\pi\,x)^2\\
&\sin((-2\,x - 2\,y)\,\pi)^2 + \cos(2\,\pi\,x)^2\,\cos((-2\,x - 2\,y)\,\pi)^2 + \sin(2\,\pi\,x)^2\,\sin(2\,\pi\,y)^2 + \cos(2\,\pi\,x)^2\,\cos(2\,\pi\,y)^2\\
&-\sin(2\,\pi\,y)\,\sin((-2\,x - 2\,y)\,\pi) - \cos(2\,\pi\,y)\,\cos((-2\,x - 2\,y)\,\pi) - \sin(2\,\pi\,x)\,\sin((-2\,x - 2\,y)\,\pi)\\
&-\cos(2\,\pi\,x)\,\cos((-2\,x - 2\,y)\,\pi) - \sin(2\,\pi\,x)\,\sin(2\,\pi\,y) - \cos(2\,\pi\,x)\,\cos(2\,\pi\,y) + \sin(2\,\pi\,y)^2\\
&\sin((-2\,x - 2\,y)\,\pi)^2 + 2\,\cos(2\,\pi\,x)\,\cos(2\,\pi\,y)\,\sin(2\,\pi\,x)\,\sin((-2\,x - 2\,y)\,\pi) + 2\,\cos(2\,\pi\,x)\\
&\cos(2\,\pi\,y)\,\sin(2\,\pi\,x)\,\sin(2\,\pi\,y) + 2\,\cos(2\,\pi\,y)\,\cos((-2\,x - 2\,y)\,\pi)\,\sin(2\,\pi\,y)\,\sin((-2\,x - 2\,y)\,\pi)\\
&+2\,\sin(2\,\pi\,x)\,\sin((-2\,x - 2\,y)\,\pi)\,\cos(2\,\pi\,y)\,\cos((-2\,x - 2\,y)\,\pi) + 2\,\cos(2\,\pi\,x)\,\cos((-2\,x - 2\,y)\,\pi)\\
&\sin(2\,\pi\,y)\,\sin((-2\,x - 2\,y)\,\pi) + 2\,\cos(2\,\pi\,x)\,\cos((-2\,x - 2\,y)\,\pi)\,\sin(2\,\pi\,x)\,\sin((-2\,x - 2\,y)\,\pi)\\
&+2\,\sin(2\,\pi\,x)\,\sin(2\,\pi\,y)\,\cos(2\,\pi\,y)\,\cos((-2\,x - 2\,y)\,\pi) + 2\,\sin(2\,\pi\,x)\,\sin(2\,\pi\,y)\,\cos(2\,\pi\,x)\\
&\cos((-2\,x - 2\,y)\,\pi) + 2\,\cos(2\,\pi\,x)\,\cos(2\,\pi\,y)\,\sin(2\,\pi\,y)\,\sin((-2\,x - 2\,y)\,\pi) + 2\,\sin(2\,\pi\,x)^2\,\sin(2\,\pi\,y)\\
&\sin((-2\,x - 2\,y)\,\pi) + 2\,\sin(2\,\pi\,x)\,\sin(2\,\pi\,y)^2\,\sin((-2\,x - 2\,y)\,\pi) + 2\,\sin(2\,\pi\,x)\,\sin((-2\,x - 2\,y)\,\pi)^2\,\sin(2\,\pi\,y)).
\end{aligned}
$$

By exploiting symmetry, one can often simplify the problem:

Here, we can rewrite the function as a polynomial $\boxed{6\,z^2 - 2\,z - 1}$!

Motivating Lattice-Based Symmetries

hexagonal



quadratic

Source: Wikipedia

Motivating Lattice-Based Symmetries

rhombic

cubic

Source: Wikipedia

Motivating Lattice-Based Symmetries

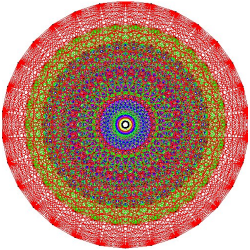


Photo credit: Matteo Fieni

Maryna Viazovska

For the proof that the E_8 lattice provides the densest packing of identical spheres in 8 dimensions, and further contributions to related extremal problems and interpolation problems in Fourier analysis.

citation | video | popular scientific exposition | CV/publications

interview | laudatio | proceedings | Plus magazine! article (intro)

Source: American Institute of Mathematics/Mathunion
Picture originally drawn by Peter McMullen BY HAND!

Content

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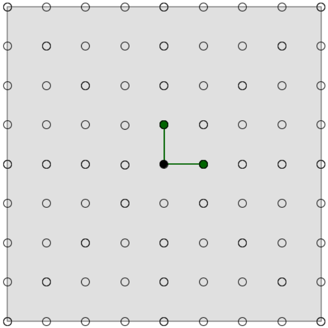
[2,4] with *Evelyne Hubert*, *Philippe Moustrou* and *Cordian Riener* and supported by European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie Actions, grant agreement 813211 (POEMA).

[3] with *Sebastian Debus*
supported by Deutsche Forschungsgemeinschaft transregional collaborative research centre (SFB–TRR) 195 "Symbolic Tools in Mathematics and their Application".

Trigonometric Polynomials with Symmetry

Lattices

Let $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n \subseteq \mathbb{R}^n$ be a lattice.



$\omega_1 = e_1, \omega_2 = e_2$

Trigonometric Polynomials

Let $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n \subseteq \mathbb{R}^n$ be a lattice and $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product.

The space of trigonometric polynomials

For $\mu \in \Omega$, define $\mathfrak{e}^\mu : \mathbb{R}^n \to \mathbb{C}$ with

$$\mathfrak{e}^\mu(u) := \exp(2\pi\mathrm{i}\,\langle \mu, u \rangle)$$

and write $\mathbb{R}[\Omega] = \mathrm{span}_{\mathbb{R}}\{\mathfrak{e}^\mu \,|\, \mu \in \Omega\}$.

$$\mathfrak{e}^\mu\,\mathfrak{e}^\nu = \mathfrak{e}^{\mu+\nu}$$
$$\mathfrak{e}^\mu\,\mathfrak{e}^{-\mu} = \mathfrak{e}^0$$

The trigonometric optimization problem

For $f = \sum_\mu f_\mu\,\mathfrak{e}^\mu \in \mathbb{R}[\Omega]$ with $f_\mu = f_{-\mu} \in \mathbb{R}$, find $f^* := \min_{u \in \mathbb{R}^n} f(u)$.

Example ($\Omega = \mathbb{Z}$)

$$f(u) = 2\,\cos(2\,\pi\,u) = \underbrace{\exp(2\,\pi\,\mathrm{i}\,u)}_{=\mathfrak{e}^1(u)} + \underbrace{\exp(-2\,\pi\,\mathrm{i}\,u)}_{=\mathfrak{e}^{-1}(u)} \quad \Rightarrow \quad f^* = -2$$

Let $\mathcal{W} \subseteq \mathrm{O}_n(\mathbb{R})$ be a finite orthogonal group w.r.t. $\langle \cdot, \cdot \rangle$ and let Ω be a **$\mathcal{W}$–lattice**, that is, for $A \in \mathcal{W}$, $\mu \in \Omega$, we have $A\,\mu \in \Omega$.

The linear action of $\mathcal{W}$ on $\mathbb{R}[\Omega]$

$$\begin{array}{ccc} \mathcal{W} \times \mathbb{R}[\Omega] & \to & \mathbb{R}[\Omega], \\ (A, \mathfrak{e}^{\mu}) & \mapsto & \mathfrak{e}^{A\,\mu} \end{array}$$

- Say f is **$\mathcal{W}$–invariant**, if $\mathcal{W} \cdot f = \{f\}$
- $\mathbb{R}[\Omega]^{\mathcal{W}}$ the **space of $\mathcal{W}$–invariants**

$$A \cdot \sum_{\mu} f_{\mu}\, \mathfrak{e}^{\mu} = \sum_{\mu} f_{\mu}\, \mathfrak{e}^{A\,\mu}$$

$$A \cdot (f\, g) = (A \cdot f)(A \cdot g)$$
$$A \cdot (f + g) = A \cdot f + A \cdot g$$

Example ($\Omega = \mathbb{Z} = -\mathbb{Z}$, $\mathcal{W} = \{\pm 1\}$)

$$f(u) := 2\cos(2\,\pi\,u) \quad \Rightarrow \quad f(u) = f(-u)$$

1st approach: Using Chebyshev Polynomials

Univariate Chebyshev Polynomials

For $\mu \in \mathbb{Z}$, define $T_\mu \in \mathbb{R}[z]$, such that

$$T_\mu(\cos(2\pi u)) = \cos(2\pi \mu u).$$

Then

$$f(u) := 2\cos(2\pi u) = 2\,T_1(\cos(2\pi u))$$

and we have

$$f^* = \min_{u\in\mathbb{R}} f(u) = \min_{z\in\mathrm{im}(\cos(2\pi u))} 2\,T_1(z) = \min_{1-z^2\geq 0} 2z = -2.$$

We require two ingredients:

1. Generalization of cosine functions and Chebyshev polynomials
2. Description of the image of the cosine functions

Generalized Chebyshev Polynomials

The generalized cosine functions

For $\mu \in \Omega$, define $\mathfrak{c}_\mu \in \mathbb{R}[\Omega]^{\mathcal{W}}$ with

$$\mathfrak{c}_\mu(u) := \frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \mathfrak{e}^{A\mu}(u).$$

$$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$$

$$\mathbb{R}[\Omega] = \mathbb{R}[\mathfrak{e}^{\pm\omega_1}, \ldots, \mathfrak{e}^{\pm\omega_n}]$$

Bourbaki's Theorem

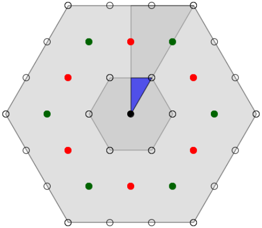
If $\mathcal{W}$ is generated by reflections, then $\mathbb{R}[\Omega]^{\mathcal{W}} = \mathbb{R}[\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}]$.

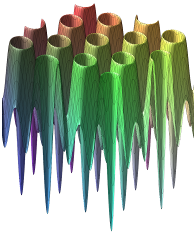
The generalized Chebyshev polynomial associated to $\mu \in \Omega$

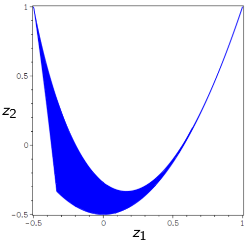
$T_\mu \in \mathbb{R}[z] = \mathbb{R}[z_1, \ldots, z_n]$, so that $T_\mu(\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) = \mathfrak{c}_\mu(u)$.

Example ($\Omega = \mathbb{Z}$)

$\mathbb{R}[\mathfrak{e}^{\pm 1}(u)]^{\{\pm 1\}} = \mathbb{R}[\cos(2\pi u)]$ and $T_\mu(\cos(2\pi u)) = \cos(2\pi \mu u)$.







Example ($\Omega = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ hexagonal lattice, $\mathcal{W} = \mathfrak{D}_{2\cdot 6}$)

For $S := \mathcal{W}\,\{2\omega_1, \omega_2\}$ and $f_{2\omega_1} := 1$, $f_{\omega_2} := 2$, we have

$$\min_{u\in\mathbb{R}^2} \sum_{\mu\in S} f_\mu\, \mathfrak{c}_\mu(u) = \min_{z\in\mathcal{T}} T_{2\omega_1}(z) + 2\, T_{\omega_2}(z) = \min_{z\in\mathcal{T}} 6\, z_1^2 - 2\, z_1 - 1 = -\frac{7}{6}$$

The new feasible region is

$$\mathcal{T} := \mathrm{im}(\mathfrak{c}) = \{\mathfrak{c}(u) := (\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) \mid u \in \mathbb{R}^n\}.$$

Appearances of $\mathcal{T}$ in the Literature

57

The transformation

(3.10) $$u=x+y,\ v=xy$$

maps the region R in the (x,y)-plane in a one-to-one and regular way onto the region $\{(u,v) \mid -v-1<u<v+1,\ u^2-4v>0\}$, which will also be denoted by R (cf. fig. 4).

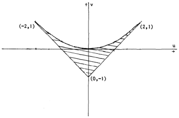


Fig. 4.

This region is bounded by two perpendicular lines and a parabola which touches the two lines. Let the weight function $\mu^{\alpha,\beta,\gamma}$ be defined by

Koornwinder'74

In [13], the existence of the Gaussian cubature rule in the sense of m-degree and the connection to orthogonal polynomials were established in the context of compact simple Lie groups. The case of the group G_2 was used as an example, where a numerical example was given. The domain $\triangle^*$ and the one in [13] differ by an affine change of variables.

Our results give explicit nodes and weights of the cubature rule and provide further explanation for the result.

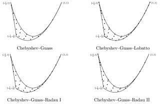


Figure 6.1. The cubature nodes on the region Δ^*.

6.2 Gauss–Lobatto cubature and Chebyshev polynomials of the first kind

In the case of $w_{-\frac{1}{2},-\frac{1}{2}}$, the change of variables $\mathbf{t} \mapsto x$ shows that (3.22) leads to a cubature of m-degree $2n-1$ based on the nodes of Y_n.

Xu'10

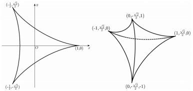


Fig. 10 The region Δ^* for $d=2$ and $d=3$

We will need the cases of $\alpha=-1/2$ and $\alpha=1/2$ of the weighted inner product

$$\langle f,g\rangle_{w^\alpha} := c_\alpha \int_{\Delta^*} f(z)\overline{g(z)}w^\alpha(z)dx,$$

where c_α is a normalization constant, $c_\alpha := 1/\int_{\Delta^*} w^\alpha(z)dx$. The change of variables $\mathbf{t} \mapsto x$ shows immediately that

Xu'12

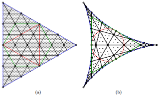


Figure 1.5 The equilateral domain Δ in (a) maps to the Deltoid δ in (b) under $t \mapsto z(t)$.

Continuous orthogonality. Let Φ be an irreducible root system on $\mathcal{V} = \mathbb{R}^d$ with an alcove $\triangle$ being the simplex defined in Lemma 1.21. The corresponding family of multivariate Chebyshev polynomials are orthogonal on the domain

Munthe–Kaas'12

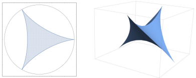


Fig. 1. The figure on the left corresponds to the orthogonality region for the case $n=2$. This is the area enclosed by Steiner's hypocycloid, which is given by an algebraic curve of fourth degree (8.2). The figure on the right is the three-dimensional region of orthogonality for $n=3$ which is determined by the algebraic equation of degree six (8.3).

$$\mathrm{vol}(\phi(A_c)) = \int_{\phi(A_c)} d\phi = \frac{(2\sqrt{\pi})^n}{\Gamma(1+\frac{n}{2})\prod_{k=1}^n \binom{n+1}{k}}.$$

For $n=2$ we obtain the area of Steiner's hypocycloid, which is $4\pi/9$. For $n=3$ we obtain the volume of the 3-dimensional analog of Steiner's hypocycloid, which equals $\pi/9$. See Fig. 1.

Koelink'20

Describing $\mathcal{T}$ as a semialgebraic set

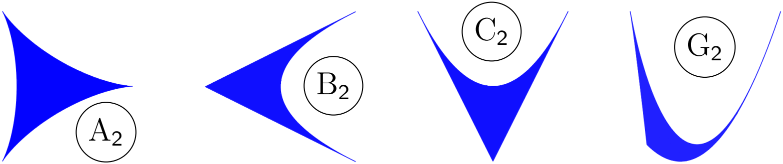
Theorem (Hubert, M, Riener'22)

For Weyl groups $\mathcal{W}$ of type A_{n-1}, B_n, C_n, D_n or G_2,
we construct a Hankel matrix polynomial $H \in \mathbb{R}[z]^{n\times n}$, such that

$$\mathcal{T} = \{z \in \mathbb{R}^n \mid H(z) \succeq 0\}$$

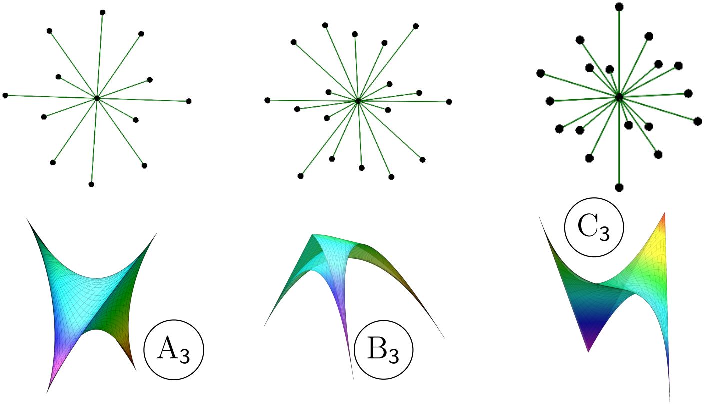
and give a closed formula in the Chebyshev basis:

$$H = \begin{pmatrix} T_0 - T_{2\omega_1} & T_{\omega_1} - T_{3\omega_1} & T_0 - T_{4\omega_1} & \cdots \\ T_{\omega_1} - T_{3\omega_1} & T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & \cdots \\ T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & 2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$



Describing $\mathcal{T}$ as a Semialgebraic Set

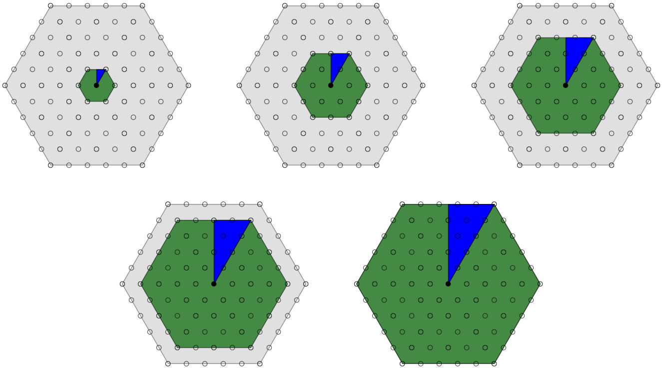
$$H = \begin{pmatrix} T_0 - T_{2\omega_1} & T_{\omega_1} - T_{3\omega_1} & T_0 - T_{4\omega_1} & \cdots \\ T_{\omega_1} - T_{3\omega_1} & T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & \cdots \\ T_0 - T_{4\omega_1} & 2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1} & 2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$



Constructing a Hierarchy of Semidefinite Bounds

We seek $f^* = \min_{z \in \mathcal{T}} \sum_{\mu} f_\mu \, T_\mu(z) = \min_{H(z) \succeq 0} \sum_{\mu} f_\mu \, T_\mu(z)$.

- (Lasserre'01) moment/sums of squares hierarchy for polynomial optimization problems with scalar constraints.
- (Henrion, Lasserre'06) ... with matrix constraints.



$$
\begin{aligned}
f^* \;=\; \min\quad & \sum_{\mu} f_\mu\, T_\mu(z) \\
\text{s.t.}\quad & z \in \mathbb{R}^n,\ H(z) \succeq 0 \\[1em]
=\; \max\quad & r \\
\text{s.t.}\quad & r \in \mathbb{R},\ \forall\, H(z) \succeq 0 : \\
& \sum_{\mu} f_\mu\, T_\mu(z) - r \geq 0.
\end{aligned}
$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

$$
\begin{aligned}
\geq\quad \sup\quad & r \\
\text{s.t.}\quad & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\
& \sum_{\mu} f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q)
\end{aligned}
$$

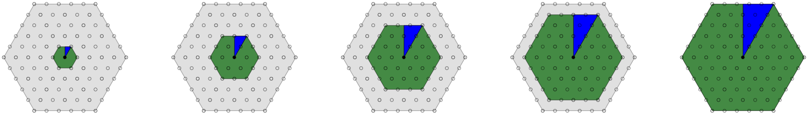
Certifying Positivity on $\mathcal{T}$

Positivstellensatz (Hol, Scherer'05)

Under the "Archimedean Assumption" on H, we have

$$\begin{aligned} f^* = \quad & \sup \quad r \\ & \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\ & \qquad \sum_{\mu} f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q). \end{aligned}$$

Denote $\Omega_d := \{\mu \in \Omega \,|\, ||\mu|| \leq d\}$.



For computations, restrict q and Q to a finite space.

Chebyshev filtration ($d \in \mathbb{N}$)

$\mathcal{F}_d := \langle T_\mu \,|\, \mu \in \Omega_d \rangle_{\mathbb{R}}$

Semidefinite Lower Bounds

SOS hierarchy for trigonometric polynomials with $\mathcal{W}$–symmetry

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$\begin{array}{rll} f^* \geq f_{\mathrm{sym}}^d := & \sup & r \\ & \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n \times n}), \\ & & \sum\limits_{\mu} f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q). \end{array}$$

Then $f_{\mathrm{sym}}^d \leq f_{\mathrm{sym}}^{d+1}$ and $\lim\limits_{d \to \infty} f_{\mathrm{sym}}^d = f^*$.

Translation to an SDP → Maple

Compute $A_0,\, A_\mu \in \mathrm{Sym}^{N(d)}$, such that

$$\begin{array}{rll} f_{\mathrm{sym}}^d = & \sup & f_0 - \mathrm{tr}(A_0\, X) \\ & \text{s.t.} & X \in \mathrm{Sym}_{\succeq 0}^{N(d)},\ \forall\, 0 \neq \mu : \\ & & \mathrm{tr}(A_\mu\, X) = f_\mu. \end{array}$$

Matrix size:

$$\begin{aligned} N(d) := & \dim(\mathcal{F}_d) \\ & + n \dim(\mathcal{F}_{d-n}) \end{aligned}$$

2nd approach: Using Symmetry Adapted Bases

Translation to Matrix Form

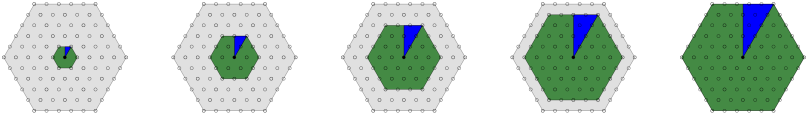
Remark

In the 1st approach, we

1. used symmetry and subsequently
2. applied a sums of squares reinforcement.

Now, we do the same thing in reverse order.

Denote $\Omega_d := \{\mu \in \Omega \,|\, ||\mu|| \leq d\}$.



If $f \in \mathbb{R}[\Omega]$ is supported on Ω_{2d}, then we can write

$$f(u) = \overline{\mathbf{E}_d(u)}^t \, \mathbf{mat}(f) \, \mathbf{E}_d(u),$$

where

1. $\mathbf{E}_d(u)$ is the vector of all $\mathfrak{e}^\mu(u)$ with $\mu \in \Omega_d$ and
2. $\mathbf{mat}(f) \in \mathbb{R}^{\Omega_d \times \Omega_d}$ is a symmetric matrix independent of u.

Translation to Matrix Form

Example

$$f := \underbrace{\mathfrak{e}^2 + \mathfrak{e}^{-2}}_{=2\cos(4\pi u)} - 2\,(\underbrace{\mathfrak{e}^1 + \mathfrak{e}^{-1}}_{=2\cos(2\pi u)}) + 3$$

$$= \underbrace{\begin{pmatrix} \mathfrak{e}^{-1} & 1 & \mathfrak{e}^1 \end{pmatrix}}_{=\overline{\mathbf{E}_1}^t} \underbrace{\begin{pmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{pmatrix}}_{=\mathbf{mat}(f)} \underbrace{\begin{pmatrix} \mathfrak{e}^1 & 1 & \mathfrak{e}^{-1} \end{pmatrix}}_{=\mathbf{E}_1}$$

is supported on $\Omega_2 = \{-2, -1, 0, 1, 2\}$.

Remark

We may always assume that $\mathbf{mat}(f)$ is a symm. Toeplitz matrix:

$$\begin{pmatrix} a & b & c \\ d & a & b \\ e & d & a \end{pmatrix} \in \mathrm{Toep}_1 \quad \xRightarrow{f_\mu = f_{-\mu}} \quad b = d,\ c = e$$

Group Action on Matrices

Semidefinite lower bounds with sums of Hermitian squares

$$f^* \geq f_d := \min_{\mathbf{X} \in \mathrm{Toep}_d} \mathrm{tr}(\mathbf{mat}(f)\,\mathbf{X}) \quad \text{s.t.} \quad \mathbf{X} \succeq 0,\ \mathrm{tr}(\mathbf{X}) = 1.$$

The action of $\mathcal{W}$ on Toeplitz matrices

$$\begin{array}{rcl} \mathcal{W} \times \mathrm{Toep}_d & \to & \mathrm{Toep}_d, \\ (A, \mathbf{X} = (\mathbf{X}_{\mu\,\nu})) & \mapsto & A \star \mathbf{X} := (\mathbf{X}_{A^{-1}\mu\, A^{-1}\nu}). \end{array}$$

We have $f \in \mathbb{R}[\Omega]^{\mathcal{W}}$ if and only if $\mathbf{mat}(f) \in \mathrm{Toep}_d^{\mathcal{W}}$.

Example ($\mathcal{W} = \{\pm 1\}$, $\Omega_1 = \{-1, 0, 1\}$)

$$\begin{array}{cc} & \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} a & b & c \\ d & a & b \\ e & d & a \end{pmatrix} \end{array} \xrightarrow{(-1) \star \ldots} \begin{array}{cc} & \begin{array}{ccc} 1 & 0 & -1 \end{array} \\ \begin{array}{c} 1 \\ 0 \\ -1 \end{array} & \begin{pmatrix} a & d & e \\ b & a & d \\ c & b & a \end{pmatrix} \end{array}$$

The action of $\mathcal{W}$ on Toeplitz matrices

$$\begin{array}{rcl} \mathcal{W} \times \mathrm{Toep}_d & \to & \mathrm{Toep}_d, \\ (A, \mathbf{X} = (\mathbf{X}_{\mu\,\nu})) & \mapsto & A \star \mathbf{X} := (\mathbf{X}_{A^{-1}\mu\, A^{-1}\nu}). \end{array}$$

We have $f \in \mathbb{R}[\Omega]^{\mathcal{W}}$ if and only if $\mathbf{mat}(f) \in \mathrm{Toep}_d^{\mathcal{W}}$.

Induced action by permutation matrices

For $A \in \mathcal{W}$, let $\vartheta(A) \in \mathrm{O}(\mathbb{R}^{\Omega_d})$ be the permutation matrix with $(\vartheta(A)\,\mathbf{x})_\mu = \mathbf{x}_{A^{-1}\mu}$ whenever $\mathbf{x} \in \mathbb{R}^{\Omega_d}$. For $\mathbf{X} \in \mathrm{Toep}_d$, we have

$$A \star \mathbf{X} = \vartheta(A)\,\mathbf{X}\,\vartheta(A)^t.$$

Isotypic decomposition

$$\mathbb{R}^{\Omega_d} = \bigoplus_{i=1}^{h} \left(\bigoplus_{j=1}^{m_i^{(d)}} V_{i\,j} \right)$$

h = number of irred. representations

$V_{i\,1} \cong \ldots \cong V_{i\,m_i^{(d)}}$ irred. ϑ–submodules

$m_i^{(d)} \in \mathbb{N}$ multiplicity

Block Diagonalization

Isotypic decomposition

$$\mathbb{R}^{\Omega_d} = \bigoplus_{i=1}^{h} \left(\bigoplus_{j=1}^{m_i^{(d)}} V_{i\,j} \right)$$

$h =$ number of irred. representations
$V_{i\,1} \cong \ldots \cong V_{i\,m_i^{(d)}}$ irred. ϑ–submodules
$m_i^{(d)} \in \mathbb{N}$ multiplicity

There is a $\mathbf{T} \in \mathrm{O}(\mathbb{R}^{\Omega_d})$ that transforms any $\mathbf{X} \in \mathrm{Toep}_d^{\mathcal{W}}$ into

$$\mathbf{T}^t\,\mathbf{X}\,\mathbf{T} = \begin{pmatrix} \boxed{\mathbf{X}_1} & & \\ & \ddots & \\ & & \boxed{\mathbf{X}_h} \end{pmatrix} \quad \text{with} \quad \mathbf{X}_i = \begin{pmatrix} \boxed{\tilde{\mathbf{X}}_i} & & \\ & \ddots & \\ & & \boxed{\tilde{\mathbf{X}}_i} \end{pmatrix},$$

$\mathbf{X}_i$ consists of $\dim(V_{i\,j})$ identical blocks $\tilde{\mathbf{X}}_i$ of size $m_i^{(d)} \times m_i^{(d)}$.

Example

Example ($\mathcal{W} = \mathfrak{S}_3$, Ω the hexagonal lattice $\subseteq \mathbb{R}^2$)

	$d=1$	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
$m_1^{(d)}$	0	1	3	6	10	15
$m_2^{(d)}$	2	6	12	20	30	42
$m_3^{(d)}$	3	6	10	15	21	28

We have $\Omega_{d=1} = \{0, -\omega_1, \omega_1 - \omega_2, \omega_2, -\omega_2, \omega_2 - \omega_1, \omega_1\}$.
The to be expected block structure is

$$\begin{pmatrix} \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} & & \\ & \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} & \\ & & \begin{bmatrix} \square & \square & \square \\ \square & \square & \square \\ \square & \square & \square \end{bmatrix} \end{pmatrix}$$

Comparison in terms of Matrix Block Sizes

Number of nonzero matrix entries (d = order of the hierarchy)

- dense: $|\Omega_d|^2$
- Chebyshev: $\frac{|\Omega_d|^2 + n^2\,|\Omega_{d-D}|^2}{|\mathcal{W}|^2}$
- Symmetry adapted basis: $\sum_{i=1}^{h} (m_i^{(d)}\, d_i)^2$

This leads to the following block sizes:

$\mathcal{W} \backslash d$	2	3	4	5	6
B_2	6, 2	10, 6	15, 12	21, 20	28, 30
	6, 3, 3, 1, 6	10, 6, 6, 3, 12	15, 10, 10, 6, 20	21, 15, 15, 10, 30	28, 21, 21, 15, 42
A_2	–	10, 3	15, 9	21, 18	28, 30
	6, 1, 6	10, 3, 12	15, 6, 20	21, 10, 30	28, 15, 42
G_2	–	6, 3	9, 6	12, 12	16, 18
	4, 2, 0, 1, 3, 3	6, 4, 1, 2, 6, 6	9, 6, 2, 4, 10, 10	12, 9, 4, 6, 15, 15	16, 12, 6, 9, 21, 21
B_3	–	13, 3	22, 9	34, 21	50, 39
	7, 3, 0, 0, 4, 1, 7, 5, 2, 1	13, 7, 1, 0, 10, 4, 17, 13, 7, 5	22, 13, 3, 1, 20, 10, 33, 27, 17, 13	34, 22, 7, 3, 35, 20, 57, 48, 33, 27	50, 34, 13, 7, 56, 35, 90, 78, 57, 48
A_3	–	–	35, 4	56, 16	84, 40
	10, 0, 5, 12, 3	20, 1, 14, 30, 12	35, 4, 30, 60, 30	56, 10, 55, 105, 60	84, 20, 91, 168, 105

first row: with Chebsyshev polynomials
second row: with symmetry adapted bases

Conclusion

Summary

- Orbit space reduction leads to a "smaller" POP and allows the use of the SOS hierarchy with Chebyshev polynomials.
  + smaller SDP
  − only works for reflection groups
- Symmetry adapted bases blockdiagonalize the SDP-matrices and preserve the numerical accuracy of the SOHS hierarchy.
  + works for any finite group
  − bigger SDP

BUT: Both techniques require a setup process (which is independent from the objective function).

Outlook

Combination of symmetry and term sparsity exploitation in POP.

Application to Spectral Bounds for Geometric Graphs

The measurable chromatic number

For $V \leqslant \mathbb{R}^n$ Abelian and $S = -S \subseteq V$ bounded with $0 \notin \overline{S}$, define

$$G(V, S) \text{ with vertices } V \text{ and } \{u, v\} \text{ an edge} \Leftrightarrow u - v \in S.$$

Problem: Find the cardinality $\chi_m(V,\ S)$ of the smallest partition of V in independent measurable sets?

Some cases of interest

- (Hardwiger, Nelson'50) $V = \mathbb{R}^2$, $S = \mathbb{S}^1$ the Euclidean sphere
- (Füredi, Kang'04) $V = \mathbb{Z}^n$, $S = \mathbb{B}^1_r$ the sphere of the 1–norm
- (Bachoc, DeCorte, de O. Filho, Vallentin'14) spectral bound
- (Bachoc, Bellito, Moustrou, Pêcher'17) polytope norms
- (Dutour Sikirić, Madore, Moustrou, Vallentin'19) lattices

Spectral Bounds

The Fourier transform of a signed Borel measure ν supported on S:

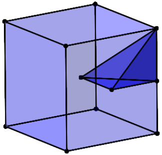
$$\hat{\nu}(u) = \int_{\mathbb{R}^n} \exp(2\pi \mathrm{i} \langle u, v \rangle) \, \mathrm{d}\nu(v).$$

Spectral bound (Bachoc, DeCorte, de Oliveira Filho, Vallentin'14)

Let $S = -S \subseteq \mathbb{R}^n$ be bounded with $0 \notin \overline{S}$ and $\mathrm{supp}(\nu) \subseteq S$. Then

$$\chi_m(\mathbb{R}^n, S) \geq 1 - \frac{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u)}{\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}.$$

Write $\widehat{\nu}$ as a trigonometric polynomial $\rightarrow$ Chebyshev polynomials

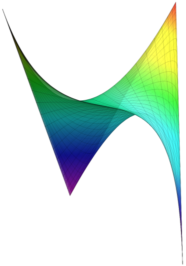


Forbidden set: $S = \partial[-1,1]^n$

Symmetry: $\mathcal{W} \cong \mathfrak{S}_n \wr \{\pm 1\}$

Discrete measure: $\nu = \sum_{i=1}^{n} \binom{n}{i} \frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \delta_{A\,\omega_i}$

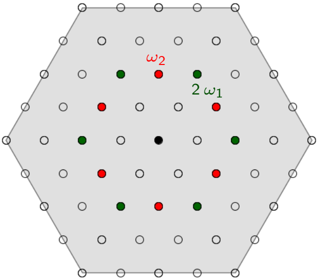
vertices and centers of
edges / faces / ... / facets

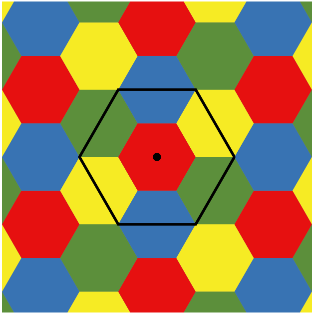


Sharp bound (with Chebyshev polynomials)

$$\chi_m(\mathbb{R}^n, \partial[-1,1]^n) = 1 - \frac{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u)}{\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)} = 2^n$$

Analytical Bound for the Hexagon





Discrete measure (as a polynomial):

$$\widehat{\nu} = T_{2\,\omega_1} + 2\,T_{\omega_2} = 6\,z_1^2 - 2\,z_1 - 1$$

Not so sharp bound (with Chebyshev polynomials)

$$4 = \chi_m(\mathbb{R}^2, \partial\text{hexagon}) \gtrsim 1 - \frac{\sup \widehat{\nu}}{\inf \widehat{\nu}} = \frac{25}{7} \approx 3.571428571$$

Restriction to Discrete Subgraphs

Spectral bound (Bachoc, DeCorte, de Oliveira Filho, Vallentin'14)

Let $S = -S \subseteq \mathbb{R}^n$ be bounded with $0 \notin \overline{S}$ and $\operatorname{supp}(\nu) \subseteq S$. Then

$$\chi_m(\mathbb{R}^n, S) \geq 1 - \frac{\sup_{u \in \mathbb{R}^n} \hat{\nu}(u)}{\inf_{u \in \mathbb{R}^n} \hat{\nu}(u)}.$$

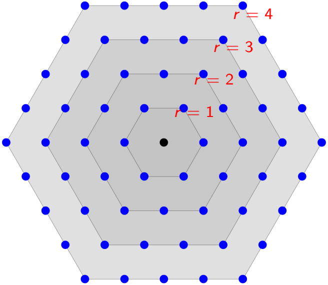
Computing the spectral bound with Chebyshev polynomials

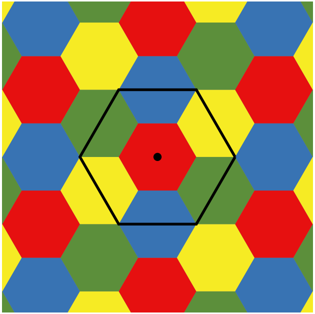
Let $S = -S = \mathcal{W}\, S \subseteq \mathbb{R}^n$ be bounded with $0 \notin \overline{S}$ and $r > 0$.
If $S_r := (r\, S) \cap \Omega \neq \emptyset$, then $\chi_m(\mathbb{R}^n, S) \geq 1 - 1/F(r)$, where

$$F(r) := \max \left\{ \min_{z \in \mathcal{T}} \sum_{\mu \in S_r^+} f_\mu \, T_\mu(z) \;\middle|\; \sum_{\mu \in S_r^+} f_\mu = 1, f_\mu \geq 0 \right\}$$

and $S_r^+ \subseteq S_r$ is minimal with $\mathcal{W}\, S_r^+ = S_r$.
Furthermore, for $k \in \mathbb{N}$, we have $F(r) \leq F(k\, r)$.

Numerical Bounds for the Hexagon





Let d be the order of the Lasserre hierarchy for $F(r)$.

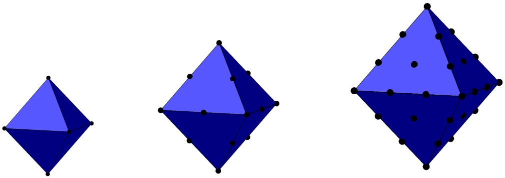
$d \backslash r$	1	2	3	4	5	6	7	8	9
3	2.99732	3.57143	3.39930	3.57143	2.47997	3.57143	–	–	–
4	2.99962	3.57143	3.52821	3.57143	3.41805	3.57143	2.54024	3.57143	–
5	3.00000	3.57143	3.52908	3.57143	3.49102	3.57143	2.76603	3.57143	2.45902
6	3.00000	3.57143	3.52912	3.57143	3.52318	3.57143	3.39290	3.57143	2.70265
7	3.00000	3.57143	3.52912	3.57143	3.54301	3.57143	3.54780	3.57143	3.53627

$r = 2$ and $\widehat{\nu} = T_{2\omega_1} + 2\,T_{\omega_2}$ seems optimal (for discrete measures).

Analytical Bounds for the 1–norm

$\chi_m(\mathbb{Z}^n, \mathbb{B}^1_r)$, where $\mathbb{B}^1_r := \{u \in \mathbb{Z}^n \mid |u_1| + \ldots + |u_n| = r\}$.

$\rightarrow$ (Füredi, Kang'04)



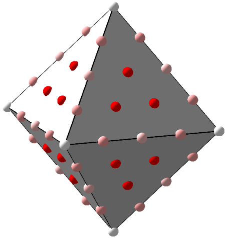
$r = 1$ $\quad$ $r = 2$ $\quad$ $r = 3$

Symmetry: $\mathcal{W} \cong \mathfrak{S}_n \wr \{\pm 1\}$

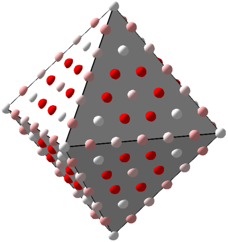
Sharp bounds for the 1–norm (with Chebyshev polynomials)

- $\chi_m(\mathbb{Z}^2, \mathbb{B}^1_{2\,r}) = 4$
- $\chi_m(\mathbb{Z}^n, \mathbb{B}^1_{2\,r+1}) = 2$
- $\chi_m(\mathbb{Z}^n, \mathbb{B}^1_2) = 2n$

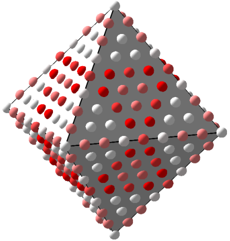
Numerical Bounds for the 1–norm ($n = 3$, $r = 4 \ldots 10$)



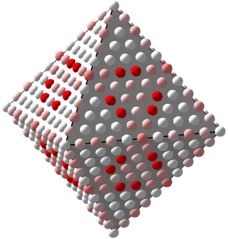
$$1 - \frac{1}{F(4)} \geq 6.28148$$



$$1 - \frac{1}{F(6)} \geq 6.30269$$



$$1 - \frac{1}{F(8)} \geq 6.30501$$



$$1 - \frac{1}{F(10)} \geq 6.30502$$

Thanks for your Attention.

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