Exploiting Term Sparsity in Symmetry-Adapted Basis for Polynomial Optimization

Igor Klep, Victor Magron, Tobias Metzlaff, Jie Wang

JNCF March 2026

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with T a finite index set and $q_t \in \mathbb{R}[X]$.

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Content

1. Historical Motivation
2. Symmetry Reduction
3. Sparsity Exploitation

Historical Motivation

1. Hilbert, 1888: Given $n, r \in \mathbb{N}$, the statement "Every nonnegative polynomial $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $2r$ is SOS." holds iff $(n, 2r) \in \{(1, 2r), (n, 2), (2, 4)\}$.

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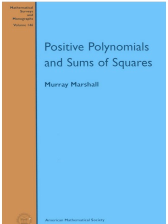

Marshall, 2008:
POSITIVE POLYNOMIALS AND SUMS OF SQUARES.
https://bookstore.ams.org/surv-146/

Polynomial Optimization

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$$f^* = \begin{array}{ll} \inf & f(X) \\ \text{s.t.} & X \in K \end{array} = \begin{array}{ll} \sup & \lambda \\ \text{s.t.} & \lambda \in \mathbb{R}, \\ & f - \lambda \geq 0 \text{ on } K \end{array} \qquad \text{(POP)}$$

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$$f^* \geq f^r_{\mathrm{sos}} := \begin{array}{ll} \sup & \lambda \\ \text{s.t.} & \lambda \in \mathbb{R}, \\ & f - \lambda \in \mathrm{QM}_r(g) \end{array}$$

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Truncated Quadratic Module

$$\mathrm{QM}_r(g) := \{q_0 + \sum_{k=0}^{\ell} q_k\, g_k \mid q_k \text{ is SOS, } \deg(q_k\, g_k) \leq 2r\}$$

Symmetry Reduction

Symmetry in Nature and Science

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Symmetry in Nature and Science

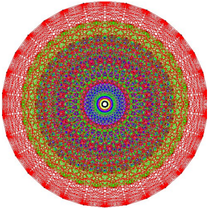
Photo credit: Matteo Fieni

Maryna Viazovska

For the proof that the E_8 lattice provides the densest packing of identical spheres in 8 dimensions, and further contributions to related extremal problems and interpolation problems in Fourier analysis.

citation | video | popular scientific exposition | CV/publications

interview | laudatio | proceedings | Plus magazine! article (intro)



Source: Wikipedia/AMS

Gosset polytope drawn BY HAND (!) by Peter McMullen, 1960s

Decomposition into Irreducibles

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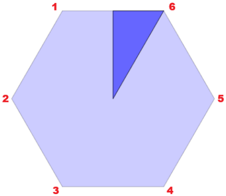
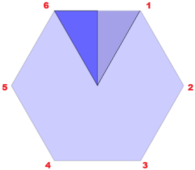
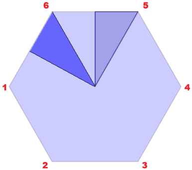


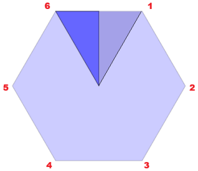


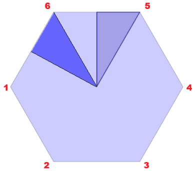


$s = (1,6)(2,5)(3,4)$ $\qquad$ $r = (1,2,3,4,5,6)$

$\mathfrak{D}_{2\cdot 6} = \langle s, r \mid s^2 = r^6 = (sr)^2 = e \rangle$ "dihedral group of order 12"

Decomposition into irreducibles





$$s = (1,6)(2,5)(3,4) \qquad r = (1,2,3,4,5,6)$$

Consider the group homomorphism $\rho : \mathfrak{D}_{2\cdot 6} \to \mathrm{GL}_6(\mathbb{R})$ with

$$\rho(s) := \begin{pmatrix} &&&&&1\\ &&&&1&\\ &&&1&&\\ &&1&&&\\ &1&&&&\\ 1&&&&& \end{pmatrix}, \quad \rho(r) := \begin{pmatrix} &1&&&&\\ &&1&&&\\ &&&1&&\\ &&&&1&\\ &&&&&1\\ 1&&&&& \end{pmatrix}.$$

Then $\mathbb{R}^6$ is $\rho(\mathfrak{D}_{2\cdot 6})$-closed and can be decomposed into

$$\mathbb{R}^6 = \left\langle \begin{pmatrix}1\\1\\1\\1\\1\\1\end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix}1\\-1\\1\\-1\\1\\-1\end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix}2\\1\\-1\\-2\\-1\\1\end{pmatrix}, \begin{pmatrix}0\\1\\1\\0\\-1\\-1\end{pmatrix} \right\rangle \oplus \left\langle \begin{pmatrix}2\\-1\\-1\\2\\-1\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\-1\\0\\1\\-1\end{pmatrix} \right\rangle.$$

Some Representation Theory

Let $\mathcal{G}$ be a finite group.

1. A $\boldsymbol{\mathcal{G}}$**-module** W is a vector space together with a group homomorphism $\rho_W : \mathcal{G} \to \mathrm{GL}(W)$, called **representation**.

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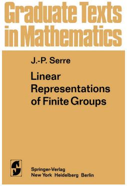

Serre, 1977:
LINEAR REPRESENTATIONS OF FINITE GROUPS.
https://link.springer.com/book/10.1007/978-1-4684-9458-7

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   _       _ _(_)_     |  Documentation: https://docs.julialang.org
  (_)     | (_) (_)    |
   _ _   _| |_  __ _   |  Type "?" for help, "]?" for Pkg help.
  | | | | | | |/ _` |  |
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 _/ |\__'_|_|_|\__'_|  |  Official https://julialang.org/ release
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julia> using Oscar
  ___   ____   ____    _    ____
 / _ \ / ___| / ___|  / \  |  _ \   |  Combining ANTIC, GAP, Polymake, Singular
| | | |\___ \| |     / _ \ | |_) |  |  Type "?Oscar" for more information
| |_| | ___) | |___ / ___ \|  _ <   |  Manual: https://docs.oscar-system.org
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julia> n=6
6

julia> G=dihedral_group(2*n)
Pc group of order 12

julia> character_table(G)
Character table of pc group of order 12

  2  2  2  1  1  2  2
  3  1  .  1  1  .  1

    1a 2a 6a 3a 2b 2c
 2P 1a 1a 3a 3a 1a 1a
 3P 1a 2a 2c 1a 2b 2c
 5P 1a 2a 6a 3a 2b 2c

X_1  1  1  1  1  1  1
X_2  1 -1 -1  1  1 -1
X_3  1 -1  1  1 -1  1
X_4  1  1 -1  1 -1 -1
X_5  2  .  1 -1  . -2
X_6  2  . -1 -1  .  2
```

Decomposition of Polynomials

Let $\mathcal{G}$ be a finite group with an induced action

$$\mathcal{G} \times \mathbb{R}[X]_r \to \mathbb{R}[X]_r, (\sigma, f) \mapsto f^\sigma.$$

E.g., if $\rho : \mathcal{G} \to \mathrm{GL}_n(\mathbb{R})$ is a rep., then $f^\sigma(X) := f(\rho(\sigma^{-1})(X))$.

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Isotypic Decomposition

$$\mathbb{R}[X]_r \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{i=1}^{h} \bigoplus_{j=1}^{m_r^{(i)}} V_j^{(i)}$$

h number of irreducible characters of $\mathcal{G}$ with multiplicities $m_r^{(i)}$ and $V_1^{(i)}, \ldots, V_{m_r^{(i)}}^{(i)}$ pairwise isomorphic irreducible $\mathcal{G}$-modules.

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Reynolds Operator

$$\mathcal{R}^{\mathcal{G}} : \mathbb{R}[X] \twoheadrightarrow \mathbb{R}[X]^{\mathcal{G}}, f \mapsto \frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} f^\sigma$$

Symmetric SOS

Observation

Let S be a basis for $\mathbb{R}[X]_r$. Any sum of squares of degree at most $2r$ can be written as

$$f = \mathbf{S}^t \cdot \mathbf{Q} \cdot \mathbf{S} \in \mathbb{R}[X]_{2r},$$

where $\mathbf{S}$ is the vector of basis elements and $\mathbf{Q} = \mathbf{Q}^t \succeq 0$ is a psd matrix of size $|S|$

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Proposition (Corollary of *Schur's Lemma*)

Let $S^{(i)} \subset \mathbb{R}[X]_r$ contain exactly one nonzero element of each $V_j^{(i)}$. If $f \in \mathbb{R}[X]^{\mathcal{G}} \cap \mathbb{R}[X]_{2r}$ is a sum of squares, then there exist $\mathbf{Q}^{(i)} = (\mathbf{Q}^{(i)})^t \succeq 0$ of size $m_r^{(i)}$, such that

$$f = \sum_{i=1}^{h} \mathcal{R}^{\mathcal{G}} \left((\mathbf{S}^{(i)})^t \cdot \mathbf{Q}^{(i)} \cdot \mathbf{S}^{(i)} \right).$$

Consequence

"The matrix of a symmetric SOS-certificate over $\mathbb{R}[X]_r$ has h blocks, each consisting of d_i many identical blocks of size $m_r^{(i)}$."

$$\dim(\mathbb{R}[X]_r) = \sum_{i=1}^{h} d_i \, m_r^{(i)} \qquad d_i := \dim(V_1^{(i)}) = \ldots = \dim(V_{m_r^{(i)}}^{(i)})$$

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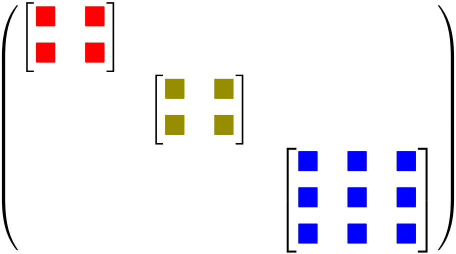

Sparsity Exploitation

What is sparsity and where does it appear?

- Correlative Sparsity:
  $f = X_1 X_2 + X_2 X_3 + \ldots + X_{99} X_{100}$
- Term Sparsity:
  $f = X_1 X_2^{99} + X_1^{99} X_2$

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- Deep learning (robustness, computer vision)
- Power systems (optimal power flow, stability)
- Quantum systems (condensed matter)

Motzkin is not SOS

Recall: $f_{\mathrm{Motzkin}} = 1 - 3\,X_1^2\,X_2^2 + X_1^4\,X_2^2 + X_1^2\,X_2^4$
is nonnegative, but **not** SOS (Motzkin, 1967).

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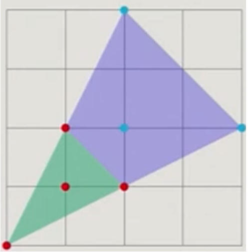
Reznick, 1978

If $f = \sum_t q_t^2$, then $\mathrm{NewtonPoly}(q_t) \subseteq \frac{1}{2}\mathrm{NewtonPoly}(f)$.



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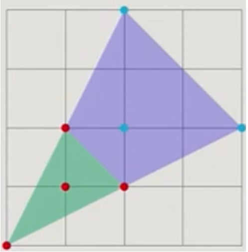


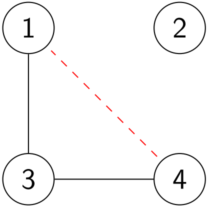
Hence, if f_{Motzkin} was SOS, then

$$f_{\mathrm{Motzkin}} = \sum (a\, 1 + \underline{b\, X_1\, X_2} + c\, X_1^2\, X_2 + d\, X_1\, X_2^2)^2$$

and thus $-3 = \sum b^2$ ☹ (see also V. Magron's slides at JNCF'23)

Encoding and Exploiting Sparsity

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & 1 & \color{red}{0} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ \color{red}{0} & 0 & 1 & 1 \end{pmatrix} \text{ with } \overline{\mathbf{B}} = \begin{pmatrix} 1 & 0 & 1 & \color{red}{1} \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ \color{red}{1} & 0 & 1 & 1 \end{pmatrix}$$



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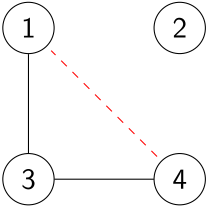


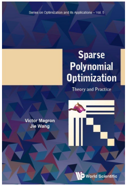
Idea: Instead of sums of squares of the form

$$f = (\mathbf{S})^t \cdot \mathbf{Q} \cdot \mathbf{S},$$

consider

$$\tilde{f} = (\mathbf{S})^t \cdot (\mathbf{B} \circ \mathbf{Q}) \cdot \mathbf{S},$$

where $\mathbf{S}$ is a vector of basis elements and $\mathbf{B}$ a binary matrix.



Magron & Wang, 2023:
SPARSE POLYNOMIAL OPTIMIZATION.
https://www.worldscientific.com/worldscibooks/10.1142/q0382

Recall:
$f_{\mathrm{Robinson}} = x_1^6 + x_2^6 - x_1^4\, x_2^2 - x_2^4\, x_1^2 - x_1^4 - x_2^4 + 3\, x_1^2\, x_2^2 - x_1^2 - x_2^2 + 1$
is nonnegative, not SOS and invariant under permutation $x_1 \leftrightarrow x_2$
(that is, invariant under the symmetric group $\mathfrak{S}_2$).

Write $f_{\mathrm{Robinson}} = a_6 - a_{42} - a_4 + 3a_{22} - a_2 + 1$ and let

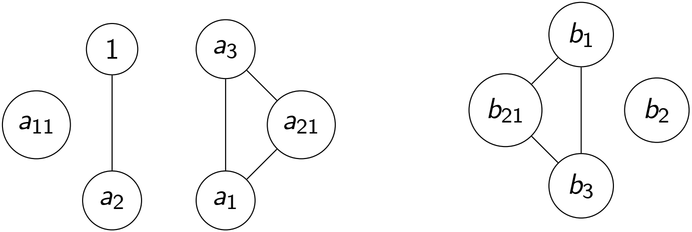
$\mathcal{S}_3^{(1)} = \{1,\ a_1 = x_1 + x_2,\ a_2 = x_1^2 + x_2^2,\ a_{11} = x_1\, x_2,\ a_3 = x_1^3 + x_2^3, a_{21} = x_1^2\, x$

$\mathcal{S}_3^{(2)} = \{b_1 = x_1 - x_2,\ b_2 = x_1^2 - x_2^2,\ b_3 = x_1^3 - x_2^3,\ b_{21} = x_1^2\, x_2 - x_1\, x_2^2\}$

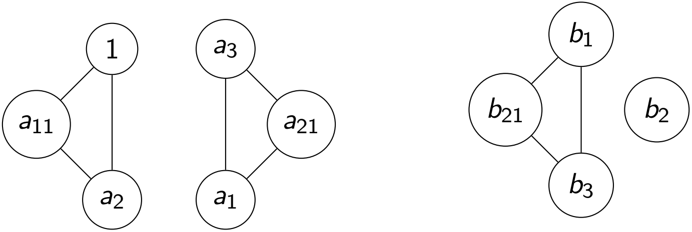
be a symmetry-adapted basis for $\mathbb{R}[X]_3$.

Sparsity Iteration for Robinson

First round of sparsity (from the data) gives the following graph:



Second round (support and chordal extension) gives:



Afterwards, the graph stabilizes.
(Note: $h = 2$ characters, but 4 components!)

Symmetry-adapted TSSOS Hierarchy

$$
\begin{aligned}
f^* = \quad \max \quad & \lambda & \geq f_{\mathrm{sos}}^{r,s} := \quad \max \quad & f_1 - \sum_{k,i} \mathrm{tr}(\mathbf{A}_{r,s,k,1}^{(i)} \cdot \mathbf{Q}_k^{(i)}) \\
\text{s.t.} \quad & \lambda \in \mathbb{R}, & \text{s.t.} \quad & \mathbf{Q}_k^{(i)} \in \mathrm{Sym}_{r-d_k}^{(i)}(\mathbf{B}_{r,s,k}^{(i)}) \\
& f - \lambda \geq 0 & & \mathbf{Q}_k^{(i)} \succeq 0, \forall j \geq 2 : \\
& \text{on } K & & f_j = \sum_{k,i} \mathrm{tr}(\mathbf{A}_{r,s,k,j}^{(i)} \cdot \mathbf{Q}_k^{(i)}),
\end{aligned}
$$

r: degree of approximation

s: level of sparsity

$\mathbf{A}_{r,s,k,j}^{(i)}$: sparse coefficient matrices in the symmetry basis

$\mathbf{B}_{r,s,k}^{(i)}$: binary matrices encoding sparsity

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\end{array}
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\text{s.t.} & \mathbf{Q}_k^{(i)} \in \mathrm{Sym}_{r-d_k}^{(i)}(\mathbf{B}_{r,s,k}^{(i)}) \\
& \mathbf{Q}_k^{(i)} \succeq 0, \forall j \geq 2 : \\
& f_j = \sum_{k,i} \operatorname{tr}(\mathbf{A}_{r,s,k,j}^{(i)} \cdot \mathbf{Q}_k^{(i)}),
\end{array}
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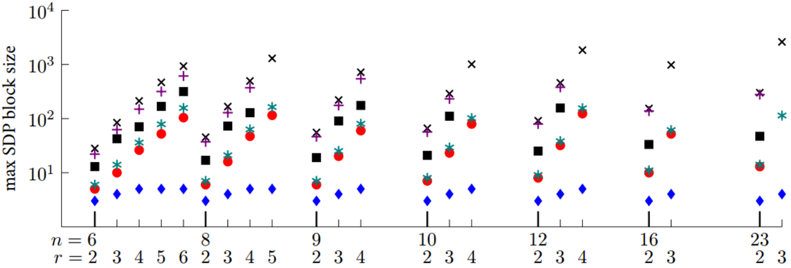
Theorem

For fixed relaxation order $r \geq r_{\min}$, the sequence $(f_{\text{sos}}^{r,s})_{s \geq 1}$ is monotonously nondecreasing with $f_{\text{sos}}^{r,*} = f_{\text{sos}}^{r}$.
For fixed sparsity order $s \geq 1$, the sequence $(f_{\text{sos}}^{r,s})_{r \geq r_{\min}}$ is monotonously nondecreasing.

For $f \in \mathbb{R}[X_1, \ldots, X_n]$ of degree $2r_{\min} = 4$ and $r \geq r_{\min}$:



× dense + sign symmetry ■ Term Sparsity ∗ Symmetry
● Symmetry + Term Sparsity 1 ♦ Symmetry + Term Sparsity 2

Thank You.

Preprint:
https://arxiv.org/abs/2511.18019
https://hal.archives-ouvertes.fr/hal-05380867

Software:
https://github.com/wangjie212/TSSOS