Spectral bounds for set avoiding graphs via polynomial optimization

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EURO2024–MC38: Applications of polynomial optimization.

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This work is supported by:

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Spectral bounds

Let G be an undirected graph with vertices V.

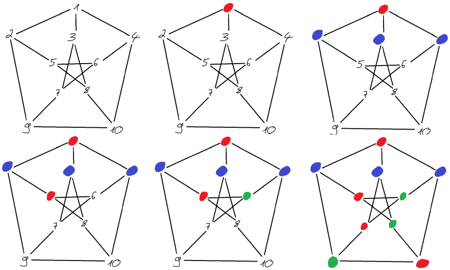
$I \subseteq V$ is called independent, if $u, v \in I \Rightarrow (u, v)$ is **not** an edge.

A coloring X of G is a partition of V in independent sets.

The chromatic number of G is

$$\chi(G) := \min\{|X|, X \text{ is a coloring of } G\}.$$

Example.



For the Petersen graph, the chromatic number is $\chi(G) = 3$.

Theorem. (Hoffman 1970)

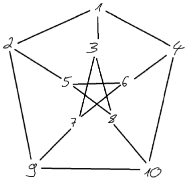
Let G be finite with adjacency matrix $A \in \{0,1\}^{V \times V}$, that is,

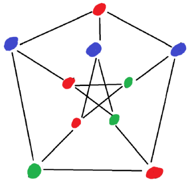
$$A_{u,v} = \begin{cases} 1, & \text{if } (u,v) \text{ is an edge,} \\ 0, & \text{otherwise.} \end{cases}$$

Since A is symmetric, all eigenvalues are real, and

$$\chi(G) \geq 1 - \frac{\lambda_{\max}(A)}{\lambda_{\min}(A)}.$$

Example.





$$\begin{pmatrix}
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0
\end{pmatrix}$$

For the Petersen graph, the eigenvalues are $3, 1, -2$ and thus

$$\chi(G) \geq 1 - \frac{3}{-2} = 2.5.$$

Set avoiding graphs

We proceed from finite to infinite graphs.

Let $S \subseteq \mathbb{R}^n$ be bounded centrally-symmetric with $0 \notin \overline{S}$.

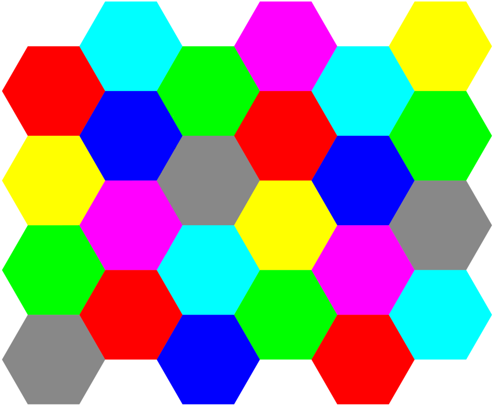
The set avoiding graph $G(S)$ is the graph with vertices $\mathbb{R}^n$ and edges (u, v), where $u - v \in S$.

A partition X of $\mathbb{R}^n$ in independent *Lebesgue–measurable* sets is called a measurable coloring of G.

The measurable chromatic number of $G(S)$ is

$$\chi_m(S) := \chi_m(G(S)) := \min\{|X|, X \text{ is a measurable coloring of } G(S)\}.$$

Example. (Hadwiger, Nelson 1950)
Consider $S = \mathbb{S}^1 = \{u \in \mathbb{R}^2 \mid \|u\| = 1\}$.



Currently, the best known lower bound (found by de Grey in 2018) and upper bound (given by the hexagonal tiling above) are

$$7 \geq \chi_m(\mathbb{S}^1) \geq 5.$$

Theorem. (Bachoc, Decorte, de Oliveira Filho, Vallentin 2014)
Let β be a positive Borel measure with support $\mathrm{supp}(\beta) \subseteq \overline{S}$ and Fourier transformation $\widehat{\beta} : \mathbb{R}^n \to \mathbb{R}$. Then

$$\chi_m(S) \geq 1 - \frac{\max_{u \in \mathbb{R}^n} \widehat{\beta}(u)}{\min_{u \in \mathbb{R}^n} \widehat{\beta}(u)}.$$

Example.
Let $S = \hexagon \subseteq \mathbb{R}^2$ be the boundary of a *regular hexagon* so that

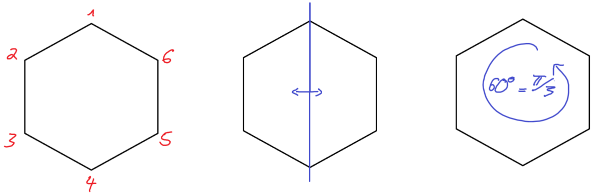
$$\mathrm{supp}(\beta) = \mathrm{vertices(hexagon)} =: \{v_1, v_2, v_3, v_4, v_5, v_6\}.$$

Then the Fourier transformation of β is

$$\widehat{\beta}(u) = \int_{\hexagon} e^{-2\pi i \langle u, v \rangle} d\beta(v) = \beta_1 e^{-2\pi i \langle u, v_1 \rangle} + \ldots + \beta_6 e^{-2\pi i \langle u, v_6 \rangle},$$

that is, a trigonometric polynomial with coefficients β_i.

Example.



$$\chi_m(\hexagon) \geq 1 - \frac{\max\limits_{u \in \mathbb{R}^2} \beta_1\, \mathrm{e}^{-2\pi \mathrm{i} \langle u, v_1 \rangle} + \ldots + \beta_6\, \mathrm{e}^{-2\pi \mathrm{i} \langle u, v_6 \rangle}}{\min\limits_{u \in \mathbb{R}^2} \beta_1\, \mathrm{e}^{-2\pi \mathrm{i} \langle u, v_1 \rangle} + \ldots + \beta_6\, \mathrm{e}^{-2\pi \mathrm{i} \langle u, v_6 \rangle}}.$$

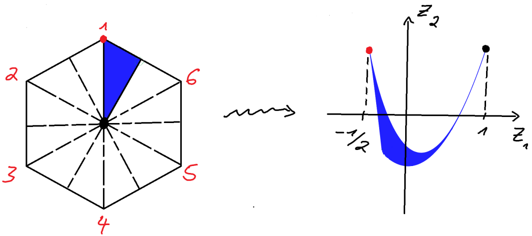
The symmetry group of $\hexagon$ is $\mathcal{W} := \mathfrak{D}_6$ (reflection + rotation).
The vertices form an orbit, that is, $\mathcal{W} \cdot v_1 = \{v_1, v_2, v_3, v_4, v_5, v_6\}$.

Theorem. (Bourbaki 1965)
If $v_i \in \mathcal{W} \cdot v_j \Rightarrow \beta_i = \beta_j$, then the Fourier transformation $\widehat{\beta}(u)$ can be written as a classical polynomial $\widehat{\beta}(z)$ on the orbit space $z(\mathbb{R}^n)$.

Theorem. (Bourbaki 1965)
If $v_i \in \mathcal{W} \cdot v_j \Rightarrow \beta_i = \beta_j$, then the Fourier transformation $\widehat{\beta}(u)$ can be written as a classical polynomial $\widehat{\beta}(z)$ on the orbit space $z(\mathbb{R}^n)$.

Theorem. (Procesi, Schwarz 1985; Hubert, M, Riener 2021)
The orbit space $z(\mathbb{R}^n)$ is a compact basic semi-algebraic set.

Example.

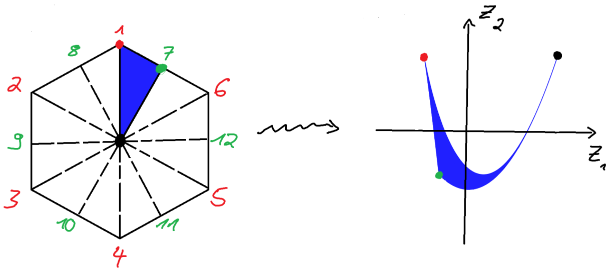


$$\chi_m(\hexagon) \geq 1 - \frac{\max\limits_{u \in \mathbb{R}^2} \widehat{\beta}(u)}{\min\limits_{u \in \mathbb{R}^2} \widehat{\beta}(u)} = 1 - \frac{\max\limits_{z_1 \in [-1/2, 1]} \beta_1 z_1}{\min\limits_{z_1 \in [-1/2, 1]} \beta_1 z_1} = 1 - \frac{1}{-1/2} = 3.$$

Example.

$$\chi_m(\hexagon) \geq 3.$$

To improve the bound, we add more boundary points.



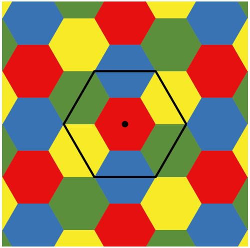
Let $\operatorname{supp}(\beta) = \mathfrak{D}_6 \cdot v_1 \cup \mathfrak{D}_6 \cdot v_7$ and $\beta_i = 1/3$, $\beta_j = 2/3$. Then

$$\chi_m(\hexagon) \geq 1 - \frac{\max_z \beta_i\,(6z_1^2 - 2z_1 - 2z_2 - 1) + \beta_j z_2}{\min_z \beta_i\,(6z_1^2 - 2z_1 - 2z_2 - 1) + \beta_j z_2} \simeq 3.57.$$

Example.

$$\chi_m(\hexagon) \geq 1 - \frac{\max_z \beta_i \left(6z_1^2 - 2z_1 - 2z_2 - 1\right) + \beta_j z_2}{\min_z \beta_i \left(6z_1^2 - 2z_1 - 2z_2 - 1\right) + \beta_j z_2} \simeq 3.57.$$

Hence, at least 4 colors are required for the graph avoiding $\hexagon$. This is indeed sufficient.



Computational aspects

Question: How to determine optimal coefficients $\beta_i, \beta_j, \ldots$?

Lemma.

We have $\max_z \widehat{\beta}(z) = \sum_i \beta_i$.

Corollary.

We have $\chi_m(S) \geq 1 - \frac{1}{F(r)}$, where

$$F(r) := \max_{\sum_i \beta_i = 1} \min_z \widehat{\beta}(z).$$

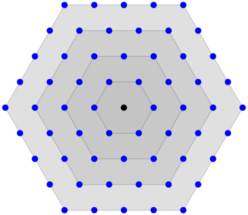
Here, the maximum is taken over all positive Borel measures β, which are supported on r orbits $\mathcal{W} \cdot v_i \subseteq S$.

Computing $F(r)$ is a max–min polynomial optimization problem.

Answer: We use a *Lasserre hierarchy*
$F(r,1) \leq F(r,2) \leq \ldots \leq F(r,d) \to F(r)$ for $d \to \infty$.
Asymptotic convergence in d happens in polynomial time.
Finite convergence in d can be certified with *flat extension*.

Question: How to certify convergence in r?
(More precisely, when is $\chi_m(S) = 1 - \frac{1}{F(r,d)}$ for $r, d \to \infty$?)

$d \backslash r$	1	2	3	4	5	6	7	8	9	10	11
3	2.99386	3.57143	3.52451	3.57143	3.37484	3.57143	–	–	–	–	–
4	3.00000	3.57143	3.52911	3.57143	3.54698	3.57143	3.47461	3.57143	–	–	–
5	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.54016	3.57143	3.51384	3.57143	–
6	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.54786	3.57143	3.55920	3.57143	3.47623
7	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.55183	3.57143	3.55921	3.57143	3.51433
8	3.00000	3.57143	3.52912	3.57143	3.54789	3.57143	3.55347	3.57143	3.55921	3.57143	3.53571



Answer: It is an open problem.
In general, $1 - \frac{1}{F(r)} \to \chi_m(S)$ is NOT true for $r \to \infty$.

Question: How to rewrite $\widehat{\beta}(u)$ as $\widehat{\beta}(z(u))$?

```
> Type,n := 'G',2; #type, dimension
  f := TPoly(Type,[2,0]) + 2*TPoly(Type,[0,1]);
  #objective function in polynomial Chebyshev basis
```

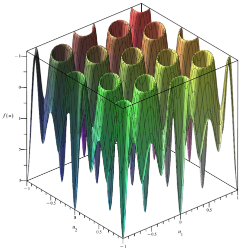
$$Type, n := G, 2$$
$$f := 6\,z_1^2 - 2\,z_1 - 1 \tag{3.1}$$

```
> GenCos:=GeneralizedCosine(Type,n,[u[1],u[2],-u[1]-u[2]]); #generalized cosines
  F:=expand(subs([seq(z[i]=GenCos[i],i=1..n)],f)); #objective function as an invariant trigonometric polynomial
```

$$GenCos := \left[\frac{1}{3}\cos(2\pi(u_1-u_2)) + \frac{1}{3}\cos(2\pi(2u_1+u_2)) + \frac{1}{3}\cos(2\pi(2u_2+u_1)), \frac{1}{3}\cos(6\pi u_2) + \frac{1}{3}\cos(2\pi(3u_1+3u_2)) + \frac{1}{3}\cos(6u_1\pi)\right]$$

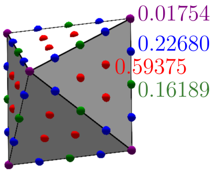
$$\begin{aligned}
F := {} & \frac{1}{3} - \frac{256}{3}\cos(u_1\pi)^6 - \frac{256}{3}\cos(\pi u_2)^6 + \frac{128}{3}\cos(u_1\pi)^8 + \frac{128}{3}\cos(\pi u_2)^8 + \frac{176}{3}\cos(\pi u_2)^4 + \frac{176}{3}\cos(u_1\pi)^4 \\
& + \frac{2848}{3}\cos(u_1\pi)^4\cos(\pi u_2)^4 - 768\cos(u_1\pi)^6\cos(\pi u_2)^4 + \frac{1408}{3}\cos(u_1\pi)^6\cos(\pi u_2)^2 - 768\cos(u_1\pi)^4\cos(\pi u_2)^6 \\
& + \frac{1408}{3}\cos(u_1\pi)^2\cos(\pi u_2)^6 + \frac{512}{3}\cos(u_1\pi)^8\cos(\pi u_2)^4 - \frac{512}{3}\cos(u_1\pi)^8\cos(\pi u_2)^2 + \frac{1024}{3}\cos(u_1\pi)^6\cos(\pi u_2)^6 \\
& + \frac{512}{3}\cos(u_1\pi)^4\cos(\pi u_2)^8 - \frac{512}{3}\cos(u_1\pi)^2\cos(\pi u_2)^8 - 416\cos(u_1\pi)^4\cos(\pi u_2)^2 - 416\cos(u_1\pi)^2\cos(\pi u_2)^4 - 16\cos(u_1\pi)^2 \\
& - 16\cos(\pi u_2)^2 + \frac{424}{3}\cos(u_1\pi)^2\cos(\pi u_2)^2 + \frac{512}{3}\sin(u_1\pi)^2\cos(u_1\pi)^2\sin(\pi u_2)^2\cos(\pi u_2)^6 \\
& - 40\sin(u_1\pi)\cos(u_1\pi)\sin(\pi u_2)\cos(\pi u_2) + \frac{736}{3}\sin(u_1\pi)\cos(u_1\pi)^3\sin(\pi u_2)\cos(\pi u_2) \\
& + \frac{736}{3}\sin(u_1\pi)\cos(u_1\pi)\sin(\pi u_2)\cos(\pi u_2)^3 - 1088\cos(u_1\pi)^3\cos(\pi u_2)^3\sin(u_1\pi)\sin(\pi u_2) \\
& + \frac{3584}{3}\cos(u_1\pi)^5\cos(\pi u_2)^3\sin(u_1\pi)\sin(\pi u_2) + \frac{3584}{3}\cos(u_1\pi)^3\cos(\pi u_2)^5\sin(u_1\pi)\sin(\pi u_2) \\
& - 384\cos(u_1\pi)^5\sin(u_1\pi)\sin(\pi u_2)\cos(\pi u_2) - 384\cos(\pi u_2)^5\sin(u_1\pi)\cos(u_1\pi)\sin(\pi u_2) \\
& + \frac{800}{3}\sin(u_1\pi)^2\cos(u_1\pi)^2\sin(\pi u_2)^2\cos(\pi u_2)^2 - \frac{1280}{3}\sin(u_1\pi)^2\cos(u_1\pi)^4\sin(\pi u_2)^2\cos(\pi u_2)^2 \\
& - \frac{1280}{3}\sin(u_1\pi)^2\cos(u_1\pi)^2\sin(\pi u_2)^2\cos(\pi u_2)^4 - \frac{1024}{3}\cos(u_1\pi)^7\cos(\pi u_2)^3\sin(u_1\pi)\sin(\pi u_2) \\
& - \frac{2048}{3}\cos(u_1\pi)^5\cos(\pi u_2)^5\sin(u_1\pi)\sin(\pi u_2) + \frac{512}{3}\cos(u_1\pi)^7\sin(u_1\pi)\sin(\pi u_2)\cos(\pi u_2) \\
& + \frac{512}{3}\sin(u_1\pi)^2\cos(u_1\pi)^6\sin(\pi u_2)^2\cos(\pi u_2)^2 + \frac{1024}{3}\sin(u_1\pi)^2\cos(u_1\pi)^4\sin(\pi u_2)^2\cos(\pi u_2)^4 \\
& - \frac{1024}{3}\cos(u_1\pi)^3\cos(\pi u_2)^7\sin(u_1\pi)\sin(\pi u_2) + \frac{512}{3}\cos(\pi u_2)^7\sin(u_1\pi)\cos(u_1\pi)\sin(\pi u_2)
\end{aligned} \tag{3.2}$$

```
> plot3d([F],u[1]=-1..1,u[2]=-1..1,grid=[200,200],
  orientation=[130,-60,180],
  labelfont=["TimesNewRoman", 15],
  labels=[u[1],u[2],'f(u)'],
  size=[1000,1000]); #plot of the trigonometric polynomial
```

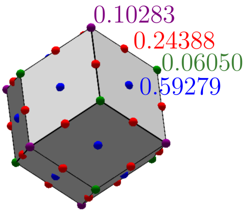


Answer: In theory, apply multiplicative invariant theory.
In practice, the procedure is implemented https://github.com/TobiasMetzlaff/GeneralizedChebyshev .

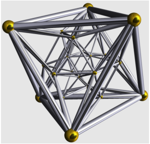
Example. (numerical bounds)



$\chi_m(\partial\,\text{octahedron}) \geq 6.28$



$\chi_m(\partial\,\text{rhombic dodecahedron}) \geq 6.12$



$\chi_m(\partial\,\text{icositetrachoron}) \geq 10.02$

Thanks for your attention.

☞ E. Hubert, T. Metzlaff, C. Riener (2021)
Orbit spaces of Weyl groups acting on compact tori: a unified and explicit polynomial description.
to appear in SIAGA.

☞ E. Hubert, T. Metzlaff, P. Moustrou, C. Riener (2022)
Optimization of trigonometric polynomials with crystallographic symmetry and spectral bounds for set avoiding graphs.
to appear in MAPRA.

☞ T. Metzlaff (2023)
On symmetry adapted bases in trigonometric optimization.
to appear in JSC.