Chebyshev Moments

Tobias Metzlaff
University of Kaiserslautern–Landau

based on joint work with E. Hubert, P. Moustrou, C. Riener

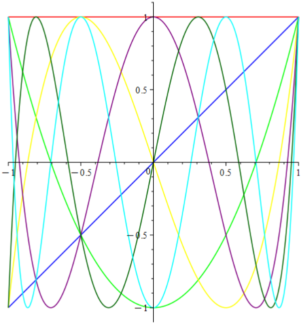
Moments and Polynomials: Applications and Theory
Konstanz, March 2024

Univariate Chebyshev polynomials (examples)

For $\alpha \in \mathbb{Z}$, the equation

$$T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

defines a unique univariate polynomial $T_\alpha \in \mathbb{R}[z]$.



$T_0 = 1,$

$T_1 = z,$

$T_2 = 2z^2 - 1,$

$T_3 = 4z^3 - 3z,$

$T_4 = 8z^4 - 8z^2 + 1,$

$T_5 = 16z^5 - 20z^3 + 5z,$

$T_6 = 32z^6 - 48z^4 + 18z^2 - 1, \ldots$

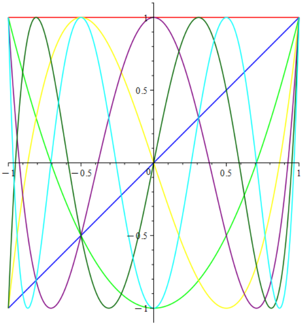
Chebyshev 1859: *Sur les questions de minima qui se rattachent a la représentation approximative des fonctions.*

Univariate Chebyshev polynomials (examples)

For $\alpha \in \mathbb{Z}$, the equation

$$T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

defines a unique univariate polynomial $T_\alpha \in \mathbb{R}[z]$.



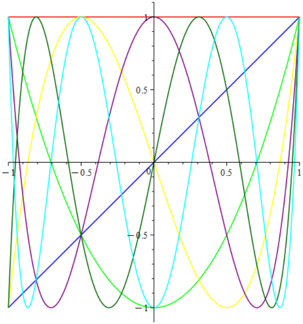
$$T_0 = 1,$$
$$T_1 = z,$$
$$T_2 = 2\,z^2 - 1,$$
$$T_3 = 4\,z^3 - 3\,z,$$
$$T_4 = 8\,z^4 - 8\,z^2 + 1,$$
$$T_5 = 16\,z^5 - 20\,z^3 + 5\,z,$$
$$T_6 = 32\,z^6 - 48\,z^4 + 18\,z^2 - 1,\ \ldots$$

Chebyshev 1859: *Sur les questions de minima qui se rattachent a la représentation approximative des fonctions*.

For $\alpha \in \mathbb{Z}$, the equation

$$T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

defines a unique univariate polynomial $T_\alpha \in \mathbb{R}[z]$.



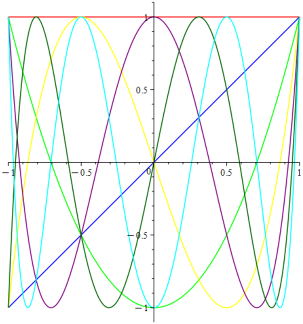
- $\int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$,
- $T_\alpha\, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$,
- $\{ T_\alpha \mid \alpha \in \mathbb{N} \}$ is a basis for $\mathbb{R}[z]$.

Univariate Chebyshev polynomials (properties)

For $\alpha \in \mathbb{Z}$, the equation

$$T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

defines a unique univariate polynomial $T_\alpha \in \mathbb{R}[z]$.



- $\int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$,
- $T_\alpha\, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$,
- $\{\, T_\alpha \mid \alpha \in \mathbb{N} \}$ is a basis for $\mathbb{R}[z]$.

Univariate Chebyshev moments

$$T_\alpha \, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$$

Let $\mathscr{L} \in \mathbb{R}[z]^*$ be a linear functional with associated operator

$$\mathbb{R}[z] \to \mathbb{R}[z]^*, \; f \mapsto \{\mathbb{R}[z] \to \mathbb{R}, \; g \mapsto \mathscr{L}(f\,g)\}.$$

In the Chebyshev basis, the matrix of this operator has entries

$$\mathbf{H}^{\mathscr{L}}_{\alpha\beta} = \mathscr{L}(T_\alpha \, T_\beta) = \frac{1}{2}\left(\mathscr{L}(T_{\alpha+\beta}) + \mathscr{L}(T_{\alpha-\beta})\right)$$

$$\mathbf{H}^{\mathscr{L}} = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & \cdots \\ y_1 & \frac{y_0}{2} + \frac{y_2}{2} & \frac{y_1}{2} + \frac{y_3}{2} & \frac{y_2}{2} + \frac{y_4}{2} & \cdots \\ y_2 & \frac{y_1}{2} + \frac{y_3}{2} & \frac{y_0}{2} + \frac{y_4}{2} & \frac{y_1}{2} + \frac{y_5}{2} & \cdots \\ y_3 & \frac{y_2}{2} + \frac{y_4}{2} & \frac{y_1}{2} + \frac{y_5}{2} & \frac{y_0}{2} + \frac{y_6}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $y_\alpha = \mathscr{L}(T_\alpha) \in \mathbb{R}$ "Chebyshev moments"

$$T_\alpha\, T_\beta = \tfrac{1}{2}\,(T_{\alpha+\beta} + T_{\alpha-\beta})$$

Let $\mathscr{L} \in \mathbb{R}[z]^*$ be a linear functional with associated operator

$$\mathbb{R}[z] \to \mathbb{R}[z]^*,\ f \mapsto \{\mathbb{R}[z] \to \mathbb{R},\ g \mapsto \mathscr{L}(f\,g)\}.$$

In the Chebyshev basis, the matrix of this operator has entries

$$\mathbf{H}^{\mathscr{L}}_{\alpha\beta} = \mathscr{L}(T_\alpha\, T_\beta) = \frac{1}{2}\,(\mathscr{L}(T_{\alpha+\beta}) + \mathscr{L}(T_{\alpha-\beta}))$$

$$\mathbf{H}^{\mathscr{L}} = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & \cdots \\ y_1 & \frac{y_0}{2}+\frac{y_2}{2} & \frac{y_1}{2}+\frac{y_3}{2} & \frac{y_2}{2}+\frac{y_4}{2} & \cdots \\ y_2 & \frac{y_1}{2}+\frac{y_3}{2} & \frac{y_0}{2}+\frac{y_4}{2} & \frac{y_1}{2}+\frac{y_5}{2} & \cdots \\ y_3 & \frac{y_2}{2}+\frac{y_4}{2} & \frac{y_1}{2}+\frac{y_5}{2} & \frac{y_0}{2}+\frac{y_6}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $y_\alpha = \mathscr{L}(T_\alpha) \in \mathbb{R}$
"Chebyshev moments"

Univariate Chebyshev moments

$$T_\alpha\, T_\beta = \tfrac{1}{2}\,(T_{\alpha+\beta} + T_{\alpha-\beta})$$

Let $\mathscr{L} \in \mathbb{R}[z]^*$ be a linear functional with associated operator

$$\mathbb{R}[z] \to \mathbb{R}[z]^*,\ f \mapsto \{\mathbb{R}[z] \to \mathbb{R},\ g \mapsto \mathscr{L}(f\,g)\}.$$

In the Chebyshev basis, the matrix of this operator has entries

$$\mathbf{H}^{\mathscr{L}}_{\alpha\beta} = \mathscr{L}(T_\alpha\, T_\beta) = \frac{1}{2}\,(\mathscr{L}(T_{\alpha+\beta}) + \mathscr{L}(T_{\alpha-\beta}))$$

$$\mathbf{H}^{\mathscr{L}} = \begin{pmatrix} y_0 & y_1 & y_2 & y_3 & \cdots \\ y_1 & \frac{y_0}{2}+\frac{y_2}{2} & \frac{y_1}{2}+\frac{y_3}{2} & \frac{y_2}{2}+\frac{y_4}{2} & \cdots \\ y_2 & \frac{y_1}{2}+\frac{y_3}{2} & \frac{y_0}{2}+\frac{y_4}{2} & \frac{y_1}{2}+\frac{y_5}{2} & \cdots \\ y_3 & \frac{y_2}{2}+\frac{y_4}{2} & \frac{y_1}{2}+\frac{y_5}{2} & \frac{y_0}{2}+\frac{y_6}{2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

with $y_\alpha = \mathscr{L}(T_\alpha) \in \mathbb{R}$
"Chebyshev moments"

Multivariate Chebyshev polynomials

Let $n \in \mathbb{N}$, $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite and $\mathbb{R}[x^\pm] := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$\mathcal{W} \times \mathbb{R}[x^\pm] \to \mathbb{R}[x^\pm]$, $(B, x^\alpha) \mapsto B \cdot x^\alpha := x^{B\alpha}$

$$\theta_i := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{Be_i}$$

"orbit polynomials"

Theorem (Bourbaki'68, Farkas'75, Steinberg'84)

$$\mathbb{R}[x^\pm]^{\mathcal{W}} = \mathbb{R}[\theta_1, \ldots, \theta_n]$$

Definition

For $\alpha \in \mathbb{Z}^n$, the equation

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$$

defines a unique multivariate polynomial $T_\alpha \in \mathbb{R}[z_1, \ldots, z_n]$.

$$T_{02} = 3\,z_2^2 - 2\,z_1,\ T_{11} = 3/2\,z_1\,z_2 - 1/2,\ T_{20} = 3\,z_1^2 - 2\,z_2,\ T_{03} = 9\,z_2^3 - 9\,z_1\,z_2 + 1,$$
$$T_{12} = 9/2\,z_1\,z_2^2 - 3\,z_1^2 - 1/2\,z_2,\ T_{21} = -3\,z_2^2 - 1/2\,z_1 + 9/2\,z_1^2\,z_2,\ T_{30} = 9\,z_1^3 - 9\,z_1\,z_2 + 1,\ \ldots$$

Multivariate Chebyshev polynomials

Let $n \in \mathbb{N}$, $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite and $\mathbb{R}[x^\pm] := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$\mathcal{W} \times \mathbb{R}[x^\pm] \to \mathbb{R}[x^\pm]$, $(B, x^\alpha) \mapsto B \cdot x^\alpha := x^{B\alpha}$

$$\theta_i := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{Be_i}$$

"orbit polynomials"

Theorem (Bourbaki'68, Farkas'75, Steinberg'84)

$$\mathbb{R}[x^\pm]^{\mathcal{W}} = \mathbb{R}[\theta_1, \ldots, \theta_n]$$

Definition

For $\alpha \in \mathbb{Z}^n$, the equation

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$$

defines a unique multivariate polynomial $T_\alpha \in \mathbb{R}[z_1, \ldots, z_n]$.

$$T_{02} = 3\,z_2^2 - 2\,z_1,\ T_{11} = 3/2\,z_1\,z_2 - 1/2,\ T_{20} = 3\,z_1^2 - 2\,z_2,\ T_{03} = 9\,z_2^3 - 9\,z_1\,z_2 + 1,$$
$$T_{12} = 9/2\,z_1\,z_2^2 - 3\,z_1^2 - 1/2\,z_2,\ T_{21} = -3\,z_2^2 - 1/2\,z_1 + 9/2\,z_1^2\,z_2,\ T_{30} = 9\,z_1^3 - 9\,z_1\,z_2 + 1,\ \ldots$$

Let $n \in \mathbb{N}$, $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite and $\mathbb{R}[x^\pm] := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$$\mathcal{W} \times \mathbb{R}[x^\pm] \to \mathbb{R}[x^\pm],\ (B, x^\alpha) \mapsto B \cdot x^\alpha := x^{B\alpha}$$

$$\theta_i := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{Be_i}$$

"orbit polynomials"

Theorem (Bourbaki'68, Farkas'75, Steinberg'84)

$$\mathbb{R}[x^\pm]^{\mathcal{W}} = \mathbb{R}[\theta_1, \ldots, \theta_n]$$

Definition

For $\alpha \in \mathbb{Z}^n$, the equation

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$$

defines a unique multivariate polynomial $T_\alpha \in \mathbb{R}[z_1, \ldots, z_n]$.

$$T_{02} = 3\,z_2^2 - 2\,z_1,\ T_{11} = 3/2\,z_1\,z_2 - 1/2,\ T_{20} = 3\,z_1^2 - 2\,z_2,\ T_{03} = 9\,z_2^3 - 9\,z_1\,z_2 + 1,$$
$$T_{12} = 9/2\,z_1\,z_2^2 - 3\,z_1^2 - 1/2\,z_2,\ T_{21} = -3\,z_2^2 - 1/2\,z_1 + 9/2\,z_1^2\,z_2,\ T_{30} = 9\,z_1^3 - 9\,z_1\,z_2 + 1,\ \ldots$$

Multivariate Chebyshev polynomials

Let $n \in \mathbb{N}$, $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite and $\mathbb{R}[x^\pm] := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$$\mathcal{W} \times \mathbb{R}[x^\pm] \to \mathbb{R}[x^\pm],\ (B, x^\alpha) \mapsto B \cdot x^\alpha := x^{B\alpha}$$

$$\theta_i := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{Be_i}$$

"orbit polynomials"

Theorem (Bourbaki'68, Farkas'75, Steinberg'84)

$$\mathbb{R}[x^\pm]^{\mathcal{W}} = \mathbb{R}[\theta_1, \ldots, \theta_n]$$

Definition

For $\alpha \in \mathbb{Z}^n$, the equation

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$$

defines a unique multivariate polynomial $T_\alpha \in \mathbb{R}[z_1, \ldots, z_n]$.

$T_{02} = 3\,z_2^2 - 2\,z_1,\ T_{11} = 3/2\,z_1\,z_2 - 1/2,\ T_{20} = 3\,z_1^2 - 2\,z_2,\ T_{03} = 9\,z_2^3 - 9\,z_1\,z_2 + 1,$

$T_{12} = 9/2\,z_1\,z_2^2 - 3\,z_1^2 - 1/2\,z_2,\ T_{21} = -3\,z_2^2 - 1/2\,z_1 + 9/2\,z_1^2\,z_2,\ T_{30} = 9\,z_1^3 - 9\,z_1\,z_2 + 1,\ \ldots$

Multivariate Chebyshev polynomials

Let $n \in \mathbb{N}$, $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ finite and $\mathbb{R}[x^\pm] := \mathbb{R}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$.

$$\mathcal{W} \times \mathbb{R}[x^\pm] \to \mathbb{R}[x^\pm],\ (B, x^\alpha) \mapsto B \cdot x^\alpha := x^{B\alpha}$$

$$\theta_i := \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{Be_i}$$

"orbit polynomials"

Theorem (Bourbaki'68, Farkas'75, Steinberg'84)

$$\mathbb{R}[x^\pm]^{\mathcal{W}} = \mathbb{R}[\theta_1, \ldots, \theta_n]$$

Definition

For $\alpha \in \mathbb{Z}^n$, the equation

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$$

defines a unique multivariate polynomial $T_\alpha \in \mathbb{R}[z_1, \ldots, z_n]$.

$$T_{02} = 3\,z_2^2 - 2\,z_1,\ T_{11} = 3/2\,z_1\,z_2 - 1/2,\ T_{20} = 3\,z_1^2 - 2\,z_2,\ T_{03} = 9\,z_2^3 - 9\,z_1\,z_2 + 1,$$
$$T_{12} = 9/2\,z_1\,z_2^2 - 3\,z_1^2 - 1/2\,z_2,\ T_{21} = -3\,z_2^2 - 1/2\,z_1 + 9/2\,z_1^2\,z_2,\ T_{30} = 9\,z_1^3 - 9\,z_1\,z_2 + 1,\ \ldots$$

Comparison: multivariate vs. univariate

$n \in \mathbb{N}$ | $n = 1$

$\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ | $\mathcal{W} = \{-1, 1\}$

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha} \qquad T_\alpha\left(\frac{x + x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

$$T_\alpha\, T_\beta = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} T_{\alpha + B\beta} \qquad T_\alpha\, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$$

$$\int_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}}\, \mathrm{d}z = \delta_{\alpha\beta} \qquad \int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1 - z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$$

[Hoffman,Withers'88]

Theorem [Procesi,Schwarz'86]

$\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Comparison: multivariate vs. univariate

$n \in \mathbb{N}$

$\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$

$n = 1$

$\mathcal{W} = \{-1, 1\}$

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha}$$

$$T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

$$T_\alpha\, T_\beta = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} T_{\alpha+B\beta}$$

$$T_\alpha\, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$$

$$\int_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}}\, \mathrm{d}z = \delta_{\alpha\beta}$$

[Hoffman,Withers'88]

$$\int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$$

Theorem [Procesi,Schwarz'86]

$\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Comparison: multivariate vs. univariate

$n \in \mathbb{N}$ | $n = 1$

$\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ | $\mathcal{W} = \{-1, 1\}$

$$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} x^{B\alpha} \qquad T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$$

$$T_\alpha \, T_\beta = \frac{1}{|\mathcal{W}|} \sum_{B \in \mathcal{W}} T_{\alpha + B\beta} \qquad T_\alpha \, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$$

$$\int_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}} \,\mathrm{d}z = \delta_{\alpha\beta} \qquad \int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}} \,\mathrm{d}z = \delta_{\alpha\beta}$$

[Hoffman,Withers'88]

Theorem [Procesi,Schwarz'86]

$\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Comparison: multivariate vs. univariate

Multivariate	Univariate
$n \in \mathbb{N}$	$n = 1$
$\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$	$\mathcal{W} = \{-1, 1\}$
$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{\lvert\mathcal{W}\rvert} \sum_{B \in \mathcal{W}} x^{B\alpha}$	$T_\alpha\left(\frac{x+x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$
$T_\alpha\, T_\beta = \frac{1}{\lvert\mathcal{W}\rvert} \sum_{B \in \mathcal{W}} T_{\alpha + B\beta}$	$T_\alpha\, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$
$\int_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}} \,\mathrm{d}z = \delta_{\alpha\beta}$	$\int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}} \,\mathrm{d}z = \delta_{\alpha\beta}$

[Hoffman,Withers'88]

Theorem [Procesi,Schwarz'86]

$\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Comparison: multivariate vs. univariate

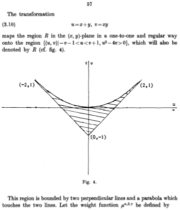
$n \in \mathbb{N}$	$n = 1$
$\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$	$\mathcal{W} = \{-1, 1\}$
$T_\alpha(\theta_1, \ldots, \theta_n) = \frac{1}{\lvert\mathcal{W}\rvert} \sum\limits_{B \in \mathcal{W}} x^{B\alpha}$	$T_\alpha\left(\frac{x + x^{-1}}{2}\right) = \frac{x^\alpha + x^{-\alpha}}{2}$
$T_\alpha\, T_\beta = \frac{1}{\lvert\mathcal{W}\rvert} \sum\limits_{B \in \mathcal{W}} T_{\alpha + B\beta}$	$T_\alpha\, T_\beta = \frac{1}{2}\left(T_{\alpha+\beta} + T_{\alpha-\beta}\right)$
$\int\limits_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}}\, \mathrm{d}z = \delta_{\alpha\beta}$	$\int\limits_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1 - z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$

[Hoffman,Withers'88]

Theorem [Procesi,Schwarz'86]

$\mathcal{T} = \{z \in \mathbb{R}^n \mid P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Orthogonality region (applications)



57

The transformation

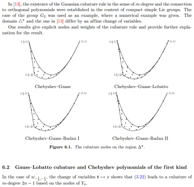
(3.10) $$u = x + y, \quad v = xy$$

maps the region R in the (x, y)-plane in a one-to-one and regular way onto the region $\{(u, v) | -v-1 < u < v+1, u^2 - 4v > 0\}$, which will also be denoted by R (cf. fig. 4).

Fig. 4.

This region is bounded by two perpendicular lines and a parabola which touches the two lines. Let the weight function $\mu^{\alpha,\beta,\gamma}$ be defined by

Koornwinder’74



In [13], the existence of the Gaussian cubature rule in the sense of m-degree and the connection to orthogonal polynomials were established in the context of compact simple Lie groups. The case of the group G_2 was used as an example, where a numerical example was given. The domain $\triangle^*$ and the one in [13] differ by an affine change of variables.

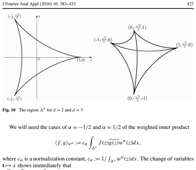
Our results give explicit nodes and weights of the cubature rule and provide further explanation for the result.

Figure 6.1. The cubature nodes on the region Δ^*.

6.2 Gauss–Lobatto cubature and Chebyshev polynomials of the first kind

In the case of $w_{-\frac{1}{2},-\frac{1}{2}}$, the change of variables $\mathbf{t} \mapsto x$ shows that (3.22) leads to a cubature of m-degree $2n-1$ based on the nodes of Y_n.

Xu’10



J Fourier Anal Appl (2010) 16: 383–433 427

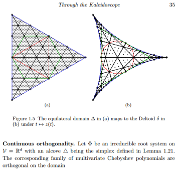
Fig. 10 The region Δ^* for $d = 2$ and $d = 3$

We will need the cases of $\alpha = -1/2$ and $\alpha = 1/2$ of the weighted inner product

$$\langle f, g \rangle_{w^\alpha} := c_\alpha \int_{\Delta^*} f(z)\overline{g(z)} w^\alpha(z) dx,$$

where c_α is a normalization constant, $c_\alpha := 1/\int_{\Delta^*} w^\alpha(z)dx$. The change of variables $\mathbf{t} \mapsto x$ shows immediately that

Xu’12

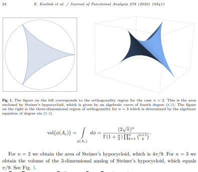


Through the Kaleidoscope 35

Figure 1.5 The equilateral domain Δ in (a) maps to the Deltoid δ in (b) under $t \mapsto z(t)$.

Continuous orthogonality. Let Φ be an irreducible root system on $\mathcal{V} = \mathbb{R}^d$ with an alcove $\triangle$ being the simplex defined in Lemma 1.21. The corresponding family of multivariate Chebyshev polynomials are orthogonal on the domain

Munthe–Kaas’12



24 *E. Koelink et al. / Journal of Functional Analysis 278 (2020) 108411*

Fig. 1. The figure on the left corresponds to the orthogonality region for the case $n = 2$. This is the area enclosed by Steiner's hypocycloid, which is given by an algebraic curve of fourth degree (8.2). The figure on the right is the three-dimensional region of orthogonality for $n = 3$ which is determined by the algebraic equation of degree six (8.3).

$$\mathrm{vol}(\phi(A_c)) = \int_{\phi(A_c)} d\phi = \frac{(2\sqrt{\pi})^n}{\Gamma(1+\frac{n}{2})\prod_{k=1}^n \binom{n+1}{k}}.$$

For $n = 2$ we obtain the area of Steiner's hypocycloid, which is $4\pi/9$. For $n = 3$ we obtain the volume of the 3-dimensional analog of Steiner's hypocycloid, which equals $\pi/9$. See Fig. 1.

Koelink’20

Orthogonality region (polynomial description)

$$\int_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}}\, \mathrm{d}z = \delta_{\alpha\beta} \qquad \int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$$

[Hoffman,Withers'88] $\quad \mathcal{T} = [-1,1],\ P(z) = 1 - z^2 = \frac{1-T_2(z)}{2}$

Theorem [Procesi,Schwarz'85]

$\mathcal{T} = \{z \in \mathbb{R}^n \,|\, P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Theorem [Hubert,M,Riener'22]

$$P = \begin{bmatrix} \frac{T_0 - T_{2\,e_1}}{4} & \frac{T_{e_1} - T_{3\,e_1}}{8} & \frac{T_0 - T_{4\,e_1}}{16} & \cdots \\ \frac{T_{e_1} - T_{3\,e_1}}{8} & \frac{T_0 - T_{4\,e_1}}{16} & \frac{2T_{e_1} - T_{3\,e_1} - T_{5\,e_1}}{32} & \cdots \\ \frac{T_0 - T_{4\,e_1}}{16} & \frac{2T_{e_1} - T_{3\,e_1} - T_{5\,e_1}}{32} & \frac{2T_0 + T_{2\,e_1} - 2T_{4\,e_1} - T_{6\,e_1}}{64} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathbb{R}[z]^{n\times n}$$

Orthogonality region (polynomial description)

$$\int_{\mathcal{T}} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{\det(P(z))}}\, \mathrm{d}z = \delta_{\alpha\beta}$$

$$\int_{-1}^{1} \frac{T_\alpha(z)\, T_\beta(z)}{\sqrt{1-z^2}}\, \mathrm{d}z = \delta_{\alpha\beta}$$

[Hoffman,Withers'88]

$\mathcal{T} = [-1,1],\ P(z) = 1 - z^2 = \frac{1-T_2(z)}{2}$

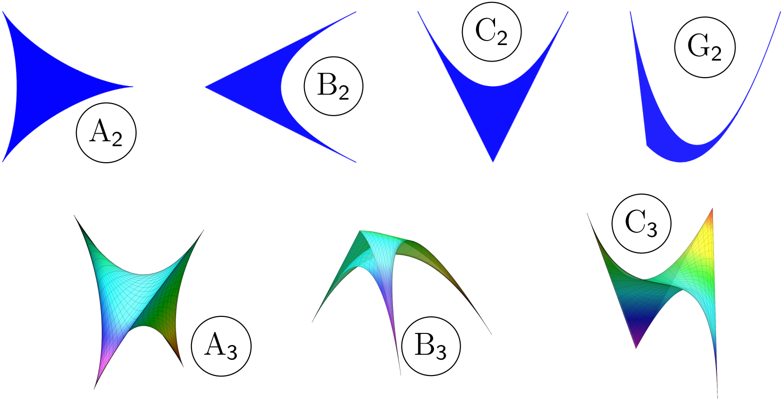
Theorem [Procesi,Schwarz'85]

$\mathcal{T} = \{z \in \mathbb{R}^n \,|\, P(z) \succeq 0\}$ is a compact basic semi–algebraic set.

Theorem [Hubert,M,Riener'22]

$$P = \begin{bmatrix} \frac{T_0 - T_{2e_1}}{4} & \frac{T_{e_1} - T_{3e_1}}{8} & \frac{T_0 - T_{4e_1}}{16} & \cdots \\ \frac{T_{e_1} - T_{3e_1}}{8} & \frac{T_0 - T_{4e_1}}{16} & \frac{2T_{e_1} - T_{3e_1} - T_{5e_1}}{32} & \cdots \\ \frac{T_0 - T_{4e_1}}{16} & \frac{2T_{e_1} - T_{3e_1} - T_{5e_1}}{32} & \frac{2T_0 + T_{2e_1} - 2T_{4e_1} - T_{6e_1}}{64} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \in \mathbb{R}[z]^{n\times n}$$

Orthogonality region (examples)



Laurent polynomial optimization

Let $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ and $f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \, x^\alpha \in \mathbb{R}[x^\pm]^{\mathcal{W}}$ with $f_\alpha = f_{-\alpha}$.

Proposition

Let $f(x) = g(\theta_1(x), \ldots, \theta_n(x))$. Then $g = \sum_{\alpha \in \mathbb{N}^n} |\mathcal{W}\alpha| \, f_\alpha \, T_\alpha$ and

$$f^* := \min_{x \in \mathbb{C}^n, \, |x_i| = 1} f(x) = \min_{z \in \mathbb{R}^n, \, P(z) \succeq 0} g(z).$$

Example ($n = 2$, $\mathcal{W} \cong \mathfrak{S}_2 \ltimes \{\pm 1\}^2$)

$$f(x) = x^{\pm e_1} + x^{\pm(e_1 - e_2)} + \frac{1}{2}\left(x^{\pm e_2} + x^{\pm(2\,e_1 - e_2)} - x^{\pm 2\,e_2} - x^{\pm(4\,e_1 - 2\,e_2)}\right)$$
$$- \frac{3}{4}\left(x^{\pm(e_1 + e_2)} + x^{\pm(3\,e_1 - e_2)} - x^{\pm(-e_1 + 2\,e_2)} - x^{\pm(3\,e_1 - 2\,e_2)}\right)$$

$$\begin{aligned} g(z) &= 4\,T_{e_1}(z) + 2\,(T_{e_2}(z) - T_{2\,e_2}(z)) - 2\,T_{e_1 + e_2}(z) \\ &= 16\,z_1^2 - 12\,z_1\,z_2 - 8\,z_2^2 + 10\,z_1 - 6\,z_2 - 2 \end{aligned}$$

Let $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ and $f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \, x^\alpha \in \mathbb{R}[x^\pm]^{\mathcal{W}}$ with $f_\alpha = f_{-\alpha}$.

Proposition

Let $f(x) = g(\theta_1(x), \ldots, \theta_n(x))$. Then $g = \sum_{\alpha \in \mathbb{N}^n} |\mathcal{W}\alpha| \, f_\alpha \, T_\alpha$ and

$$f^* := \min_{x \in \mathbb{C}^n, \, |x_i|=1} f(x) = \min_{z \in \mathbb{R}^n, \, P(z) \succeq 0} g(z).$$

Example ($n = 2$, $\mathcal{W} \cong \mathfrak{S}_2 \ltimes \{\pm 1\}^2$)

$$f(x) = x^{\pm e_1} + x^{\pm(e_1 - e_2)} + \frac{1}{2}\left(x^{\pm e_2} + x^{\pm(2\,e_1 - e_2)} - x^{\pm 2\,e_2} - x^{\pm(4\,e_1 - 2\,e_2)}\right)$$
$$- \frac{3}{4}\left(x^{\pm(e_1 + e_2)} + x^{\pm(3\,e_1 - e_2)} - x^{\pm(-e_1 + 2\,e_2)} - x^{\pm(3\,e_1 - 2\,e_2)}\right)$$

$$g(z) = 4\,T_{e_1}(z) + 2\,(T_{e_2}(z) - T_{2\,e_2}(z)) - 2\,T_{e_1+e_2}(z)$$
$$= 16\,z_1^2 - 12\,z_1\,z_2 - 8\,z_2^2 + 10\,z_1 - 6\,z_2 - 2$$

Laurent polynomial optimization

Let $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ and $f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \, x^\alpha \in \mathbb{R}[x^\pm]^{\mathcal{W}}$ with $f_\alpha = f_{-\alpha}$.

Proposition

Let $f(x) = g(\theta_1(x), \ldots, \theta_n(x))$. Then $g = \sum_{\alpha \in \mathbb{N}^n} |\mathcal{W}\alpha| \, f_\alpha \, T_\alpha$ and

$$f^* := \min_{x \in \mathbb{C}^n, \, |x_i|=1} f(x) = \min_{z \in \mathbb{R}^n, \, P(z) \succeq 0} g(z).$$

Example ($n = 2$, $\mathcal{W} \cong \mathfrak{S}_2 \ltimes \{\pm 1\}^2$)

$$f(x) = x^{\pm e_1} + x^{\pm(e_1 - e_2)} + \frac{1}{2}\left(x^{\pm e_2} + x^{\pm(2\,e_1 - e_2)} - x^{\pm 2\,e_2} - x^{\pm(4\,e_1 - 2\,e_2)}\right) - \frac{3}{4}\left(x^{\pm(e_1+e_2)} + x^{\pm(3\,e_1 - e_2)} - x^{\pm(-e_1 + 2\,e_2)} - x^{\pm(3\,e_1 - 2\,e_2)}\right)$$

$$g(z) = 4\,T_{e_1}(z) + 2\,(T_{e_2}(z) - T_{2\,e_2}(z)) - 2\,T_{e_1+e_2}(z) = 16\,z_1^2 - 12\,z_1\,z_2 - 8\,z_2^2 + 10\,z_1 - 6\,z_2 - 2$$

Laurent polynomial optimization

Let $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ and $f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \, x^\alpha \in \mathbb{R}[x^\pm]^{\mathcal{W}}$ with $f_\alpha = f_{-\alpha}$.

Proposition

Let $f(x) = g(\theta_1(x), \ldots, \theta_n(x))$. Then $g = \sum_{\alpha \in \mathbb{N}^n} |\mathcal{W}\alpha| \, f_\alpha \, T_\alpha$ and

$$f^* := \min_{x \in \mathbb{C}^n, \, |x_i|=1} f(x) = \min_{z \in \mathbb{R}^n, \, P(z) \succeq 0} g(z).$$

[Dumitrescu'07]

Hermitian SOS for Laurent PolyOpt

[Lasserre'01]

moment/SOS for PolyOpt with scalar constraints [Putinar'93]

[Henrion, Lasserre'06]

moment/SOS for PolyOpt with matrix constraints [Hol,Scherer'05]

Laurent polynomial optimization

Let $\mathcal{W} \subseteq \mathrm{GL}_n(\mathbb{Z})$ and $f = \sum_{\alpha \in \mathbb{Z}^n} f_\alpha \, x^\alpha \in \mathbb{R}[x^\pm]^{\mathcal{W}}$ with $f_\alpha = f_{-\alpha}$.

Proposition

Let $f(x) = g(\theta_1(x), \ldots, \theta_n(x))$. Then $g = \sum_{\alpha \in \mathbb{N}^n} |\mathcal{W}\alpha| \, f_\alpha \, T_\alpha$ and

$$f^* := \min_{x \in \mathbb{C}^n, \, |x_i|=1} f(x) = \min_{z \in \mathbb{R}^n, \, P(z) \succeq 0} g(z).$$

[Dumitrescu'07]

Hermitian SOS for Laurent PolyOpt

[Lasserre'01]

moment/SOS for PolyOpt with scalar constraints [Putinar'93]

[Henrion, Lasserre'06]

moment/SOS for PolyOpt with matrix constraints [Hol,Scherer'05]

Matrix SOS reinforcement

$$
\begin{aligned}
f^* &= \min \quad \sum_\alpha f_\alpha\, T_\alpha(z) \\
&\quad \text{s.t.} \quad z \in \mathbb{R}^n,\ P(z) \succeq 0 \\
&= \max \quad r \\
&\quad \text{s.t.} \quad r \in \mathbb{R},\ \forall\, P(z) \succeq 0 : \\
&\qquad\qquad \sum_\alpha f_\alpha\, T_\alpha(z) - r \geq 0
\end{aligned}
$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

$$
\begin{aligned}
&\geq \max \quad r \\
&\quad \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\
&\qquad\qquad \sum_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q)
\end{aligned}
$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$$\mathcal{F}_d := \langle T_\alpha \mid \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_{\mathbb{R}}$$

$$T_\alpha\, T_\beta = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \alpha+\beta, \rho_0^\vee \rangle} t_\omega\, T_\omega$$

If $T_\alpha \in \mathcal{F}_{d_1}$ and $T_\beta \in \mathcal{F}_{d_2}$, then $T_\alpha\, T_\beta \in \mathcal{F}_{d_1+d_2}$.

Matrix SOS reinforcement

$$f^* = \min \sum_\alpha f_\alpha \, T_\alpha(z) \quad \text{s.t.} \quad z \in \mathbb{R}^n,\ P(z) \succeq 0$$

$$= \max \ r \quad \text{s.t.} \quad r \in \mathbb{R},\ \forall\, P(z) \succeq 0 : \sum_\alpha f_\alpha \, T_\alpha(z) - r \geq 0$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n \times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

$$\geq \max \ r \quad \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n \times n}),\ \sum_\alpha f_\alpha \, T_\alpha - r = q + \mathrm{tr}(P\, Q)$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$\mathcal{F}_d := \langle T_\alpha \mid \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$

$$T_\alpha \, T_\beta = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \alpha+\beta, \rho_0^\vee \rangle} t_\omega \, T_\omega$$

If $T_\alpha \in \mathcal{F}_{d_1}$ and $T_\beta \in \mathcal{F}_{d_2}$, then $T_\alpha \, T_\beta \in \mathcal{F}_{d_1+d_2}$.

Matrix SOS reinforcement

$$
\begin{aligned}
f^* &= \min \quad \sum_\alpha f_\alpha \, T_\alpha(z) \\
&\quad\ \text{s.t.} \quad z \in \mathbb{R}^n,\ P(z) \succeq 0 \\[1em]
&= \max \quad r \\
&\quad\ \text{s.t.} \quad r \in \mathbb{R},\ \forall\, P(z) \succeq 0 : \\
&\qquad\qquad \sum_\alpha f_\alpha \, T_\alpha(z) - r \geq 0
\end{aligned}
$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

$$
\begin{aligned}
&\geq \max \quad r \\
&\quad\ \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\
&\qquad\qquad \sum_\alpha f_\alpha \, T_\alpha - r = q + \mathrm{tr}(P\, Q)
\end{aligned}
$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$\mathcal{F}_d := \langle T_\alpha \mid \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$

$$T_\alpha \, T_\beta = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \alpha+\beta, \rho_0^\vee \rangle} t_\omega \, T_\omega$$

If $T_\alpha \in \mathcal{F}_{d_1}$ and $T_\beta \in \mathcal{F}_{d_2}$, then $T_\alpha \, T_\beta \in \mathcal{F}_{d_1+d_2}$.

Matrix SOS reinforcement

$$\begin{aligned} f^* &= \min \quad \sum_\alpha f_\alpha \, T_\alpha(z) \\ &\quad \text{s.t.} \quad z \in \mathbb{R}^n, \; P(z) \succeq 0 \end{aligned}$$

$$\begin{aligned} &= \max \quad r \\ &\quad \text{s.t.} \quad r \in \mathbb{R}, \; \forall \, P(z) \succeq 0 : \\ &\qquad\qquad \sum_\alpha f_\alpha \, T_\alpha(z) - r \geq 0 \end{aligned}$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists \, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z) \, Q_i(z)^t$$

$$\begin{aligned} &\geq \max \quad r \\ &\quad \text{s.t.} \quad r \in \mathbb{R}, \; q \in \mathrm{SOS}(\mathbb{R}[z]), \; Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\ &\qquad\qquad \sum_\alpha f_\alpha \, T_\alpha - r = q + \mathrm{tr}(P\,Q) \end{aligned}$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$$\mathcal{F}_d := \langle T_\alpha \,|\, \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_{\mathbb{R}}$$

$$T_\alpha \, T_\beta = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \alpha+\beta, \rho_0^\vee \rangle} t_\omega \, T_\omega$$

If $T_\alpha \in \mathcal{F}_{d_1}$ and $T_\beta \in \mathcal{F}_{d_2}$, then $T_\alpha \, T_\beta \in \mathcal{F}_{d_1+d_2}$.

Matrix SOS reinforcement

$$
\begin{aligned}
f^* &= \min && \sum_\alpha f_\alpha\, T_\alpha(z) \\
& \text{s.t.} && z \in \mathbb{R}^n,\ P(z) \succeq 0 \\[1em]
&= \max && r \\
& \text{s.t.} && r \in \mathbb{R},\ \forall\, P(z) \succeq 0 : \\
& && \sum_\alpha f_\alpha\, T_\alpha(z) - r \geq 0
\end{aligned}
$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^{k} Q_i(z)\, Q_i(z)^t$$

$$
\begin{aligned}
\geq\ & \max && r \\
& \text{s.t.} && r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\
& && \sum_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q)
\end{aligned}
$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$$\mathcal{F}_d := \langle T_\alpha \,|\, \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_{\mathbb{R}}$$

$$T_\alpha\, T_\beta = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \alpha+\beta, \rho_0^\vee \rangle} t_\omega\, T_\omega$$

If $T_\alpha \in \mathcal{F}_{d_1}$ and $T_\beta \in \mathcal{F}_{d_2}$, then $T_\alpha\, T_\beta \in \mathcal{F}_{d_1+d_2}$.

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\alpha \,|\, \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$f^* \geq f_{\mathrm{sos}}^d := \begin{array}{ll} \max & r \\ \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}), \\ & \sum\limits_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q). \end{array}$$

Then $f_{\mathrm{sos}}^d \leq f_{\mathrm{sos}}^{d+1}$ and $\lim\limits_{d\to\infty} f_{\mathrm{sos}}^d = f^*$.

Translation to an SDP → MAPLE[1]

Compute $A_0,\ A_\alpha \in \mathrm{Sym}^d$, such that

$$f_{\mathrm{sos}}^d = \begin{array}{ll} \max & f_0 - \mathrm{tr}(A_0\, \mathbf{X}) \\ \text{s.t.} & \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d,\ \forall\, 0 \neq \alpha : \\ & \mathrm{tr}(A_\alpha\, \mathbf{X}) = f_\alpha. \end{array}$$

Matrix size: $\dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$

[1] https://github.com/TobiasMetzlaff/GeneralizedChebyshev

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\alpha \,|\, \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$
\begin{aligned}
f^* \geq f_{\mathrm{sos}}^d := \quad & \max \quad r \\
& \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}), \\
& \qquad\ \sum_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q).
\end{aligned}
$$

Then $f_{\mathrm{sos}}^d \leq f_{\mathrm{sos}}^{d+1}$ and $\lim\limits_{d\to\infty} f_{\mathrm{sos}}^d = f^*$.

Translation to an SDP $\to$ MAPLE[1]

Compute $A_0,\, A_\alpha \in \mathrm{Sym}^d$, such that

$$
\begin{aligned}
f_{\mathrm{sos}}^d = \quad & \max \quad f_0 - \mathrm{tr}(A_0\, \mathbf{X}) \\
& \text{s.t.} \quad \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d,\ \forall\, 0 \neq \alpha : \\
& \qquad\ \mathrm{tr}(A_\alpha\, \mathbf{X}) = f_\alpha.
\end{aligned}
$$

Matrix size:
$\dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$

[1] https://github.com/TobiasMetzlaff/GeneralizedChebyshev

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\alpha \,|\, \langle \alpha, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$f^* \geq f_{\mathrm{sos}}^d := \begin{array}{ll} \max & r \\ \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}), \\ & \sum\limits_{\alpha} f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q). \end{array}$$

Then $f_{\mathrm{sos}}^d \leq f_{\mathrm{sos}}^{d+1}$ and $\lim\limits_{d\to\infty} f_{\mathrm{sos}}^d = f^*$.

Translation to an SDP $\to$ Maple[1]

Compute $A_0,\, A_\alpha \in \mathrm{Sym}^d$, such that

$$f_{\mathrm{sos}}^d = \begin{array}{ll} \max & f_0 - \mathrm{tr}(A_0\, \mathbf{X}) \\ \text{s.t.} & \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d,\ \forall\, 0 \neq \alpha : \\ & \mathrm{tr}(A_\alpha\, \mathbf{X}) = f_\alpha. \end{array}$$

Matrix size:
$\dim(\mathcal{F}_d) + n\, \dim(\mathcal{F}_{d-n})$

[1] https://github.com/TobiasMetzlaff/GeneralizedChebyshev

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f^d_{\mathrm{sos}} := \max \; r \quad \text{s.t.} \quad r \in \mathbb{R},\; q \in \mathrm{SOS}(\mathcal{F}_d),\; Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}),\; \sum_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q)$$

Moment relaxation for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f^d_{\mathrm{mom}} := \min \; \sum_\alpha f_\alpha\, \mathscr{L}(T_\alpha) \quad \text{s.t.} \quad \mathscr{L} \in {\mathcal{F}_{2d}}^*,\; \mathscr{L}(1) = 1,\; H^{\mathscr{L}}_d,\; H^{P*\mathscr{L}}_{d-n} \succeq 0$$

Dual SDP

$$f^d_{\mathrm{sos}} = \max \; f_0 - \mathrm{tr}(A_0\, \mathbf{X}) \quad \text{s.t.} \quad \mathbf{X} \in \mathrm{Sym}^d_{\succeq 0},\; \mathrm{tr}(A_\alpha\, \mathbf{X}) = f_\alpha$$

Primal SDP

$$f^d_{\mathrm{mom}} = \min \; \sum_\alpha f_\alpha\, \mathbf{y}_\alpha \quad \text{s.t.} \quad \mathbf{y} \in \mathbb{R}^{\dim(\mathcal{F}_{2d})},\; \mathbf{y}_0 = 1,\; \mathbf{Z} = \sum_\alpha \mathbf{y}_\alpha\, A_\alpha \in \mathrm{Sym}^d_{\succeq 0}$$

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f_{\mathrm{sos}}^d := \max \; r \quad \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}),\quad \sum_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q)$$

Moment relaxation for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f_{\mathrm{mom}}^d := \min \; \sum_\alpha f_\alpha\, \mathscr{L}(T_\alpha) \quad \text{s.t.} \quad \mathscr{L} \in {\mathcal{F}_{2d}}^*,\ \mathscr{L}(1) = 1,\ H_d^{\mathscr{L}},\ H_{d-n}^{P*\mathscr{L}} \succeq 0$$

Dual SDP

$$f_{\mathrm{sos}}^d = \max \; f_0 - \mathrm{tr}(A_0\, \mathbf{X}) \quad \text{s.t.} \quad \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d,\quad \mathrm{tr}(A_\alpha\, \mathbf{X}) = f_\alpha$$

Primal SDP

$$f_{\mathrm{mom}}^d = \min \; \sum_\alpha f_\alpha\, \mathbf{y}_\alpha \quad \text{s.t.} \quad \mathbf{y} \in \mathbb{R}^{\dim(\mathcal{F}_{2d})},\ \mathbf{y}_0 = 1,\quad \mathbf{Z} = \sum_\alpha \mathbf{y}_\alpha\, A_\alpha \in \mathrm{Sym}_{\succeq 0}^d$$

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f_{\mathrm{sos}}^d := \max \; r \quad \text{s.t.} \; r \in \mathbb{R},\; q \in \mathrm{SOS}(\mathcal{F}_d),\; Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}),\; \sum_\alpha f_\alpha T_\alpha - r = q + \mathrm{tr}(P\,Q)$$

Moment relaxation for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f_{\mathrm{mom}}^d := \min \; \sum_\alpha f_\alpha \mathscr{L}(T_\alpha) \quad \text{s.t.} \; \mathscr{L} \in {\mathcal{F}_{2d}}^*,\; \mathscr{L}(1) = 1,\; H_d^{\mathscr{L}},\; H_{d-n}^{P*\mathscr{L}} \succeq 0$$

Dual SDP

$$f_{\mathrm{sos}}^d = \max \; f_0 - \mathrm{tr}(A_0\,\mathbf{X}) \quad \text{s.t.} \; \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d,\; \mathrm{tr}(A_\alpha\,\mathbf{X}) = f_\alpha$$

Primal SDP

$$f_{\mathrm{mom}}^d = \min \; \sum_\alpha f_\alpha \mathbf{y}_\alpha \quad \text{s.t.} \; \mathbf{y} \in \mathbb{R}^{\dim(\mathcal{F}_{2d})},\; \mathbf{y}_0 = 1,\; \mathbf{Z} = \sum_\alpha \mathbf{y}_\alpha A_\alpha \in \mathrm{Sym}_{\succeq 0}^d$$

Semi–definite lower bounds

SOS reinforcement for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f_{\mathrm{sos}}^d := \begin{array}{rl} \max & r \\ \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}), \\ & \sum_\alpha f_\alpha\, T_\alpha - r = q + \mathrm{tr}(P\, Q) \end{array}$$

Moment relaxation for positive $\mathcal{W}$–invariant Laurent polynomials

$$f^* \geq f_{\mathrm{mom}}^d := \begin{array}{rl} \min & \sum_\alpha f_\alpha\, \mathscr{L}(T_\alpha) \\ \text{s.t.} & \mathscr{L} \in \mathcal{F}_{2d}{}^*,\ \mathscr{L}(1) = 1,\ H_d^{\mathscr{L}},\ H_{d-n}^{P*\mathscr{L}} \succeq 0 \end{array}$$

Dual SDP

$$f_{\mathrm{sos}}^d = \begin{array}{rl} \max & f_0 - \mathrm{tr}(A_0\, \mathbf{X}) \\ \text{s.t.} & \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d, \\ & \mathrm{tr}(A_\alpha\, \mathbf{X}) = f_\alpha \end{array}$$

Primal SDP

$$f_{\mathrm{mom}}^d = \begin{array}{rl} \min & \sum_\alpha f_\alpha\, \mathbf{y}_\alpha \\ \text{s.t.} & \mathbf{y} \in \mathbb{R}^{\dim(\mathcal{F}_{2d})},\ \mathbf{y}_0 = 1, \\ & \mathbf{Z} = \sum_\alpha \mathbf{y}_\alpha\, A_\alpha \in \mathrm{Sym}_{\succeq 0}^d \end{array}$$

Semi–definite lower bounds

Dual SDP

$$f_{\mathrm{sos}}^d = \max \quad f_0 - \mathrm{tr}(A_0\,\mathbf{X}) \quad \text{s.t.} \quad \mathbf{X} \in \mathrm{Sym}_{\succeq 0}^d, \quad \mathrm{tr}(A_\alpha\,\mathbf{X}) = f_\alpha$$

Primal SDP

$$f_{\mathrm{mom}}^d = \min \quad \sum_\alpha f_\alpha\,\mathbf{y}_\alpha \quad \text{s.t.} \quad \mathbf{y} \in \mathbb{R}^{\dim(\mathcal{F}_{2d})},\ \mathbf{y}_0 = 1, \quad \mathbf{Z} = \sum_\alpha \mathbf{y}_\alpha\,A_\alpha \in \mathrm{Sym}_{\succeq 0}^d$$

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^d = f_{\mathrm{mom}}^d = f^*$?

Semi–definite lower bounds

Dual SDP

$$f_{\mathrm{sos}}^d = \begin{array}{rl} \max & f_0 - \mathrm{tr}(A_0\,\mathbf{X}) \\ \text{s.t.} & \mathbf{X} \in \mathrm{Sym}^d_{\succeq 0}, \\ & \mathrm{tr}(A_\alpha\,\mathbf{X}) = f_\alpha \end{array}$$

Primal SDP

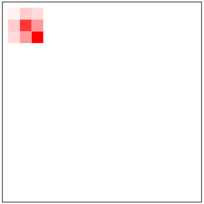
$$f_{\mathrm{mom}}^d = \begin{array}{rl} \min & \sum_\alpha f_\alpha\,\mathbf{y}_\alpha \\ \text{s.t.} & \mathbf{y} \in \mathbb{R}^{\dim(\mathcal{F}_{2d})},\ \mathbf{y}_0 = 1, \\ & \mathbf{Z} = \sum_\alpha \mathbf{y}_\alpha\,A_\alpha \in \mathrm{Sym}^d_{\succeq 0} \end{array}$$

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

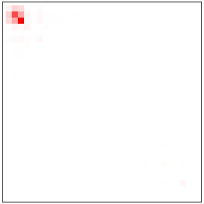
Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^d = f_{\mathrm{mom}}^d = f^*$?

Output SOS–solution ($n = 3$, $\mathcal{W} \cong \mathfrak{S}_3 \ltimes \{\pm 1\}^3$)

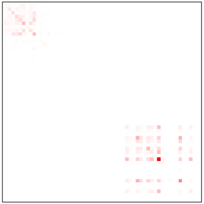
$\mathbf{X} \in \mathbb{R}^{N_d \times N_d}$ matrix of size $N_d = \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$



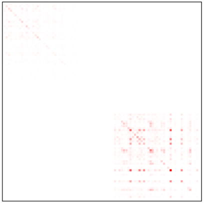
$d = 3, \; N_d = 13 + 3$



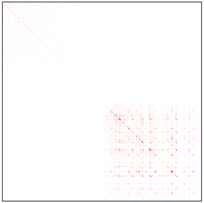
$d = 4, \; N_d = 22 + 9$



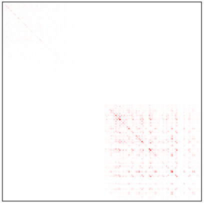
$d = 5, \; N_d = 34 + 21$



$d = 6, \; N_d = 50 + 39$

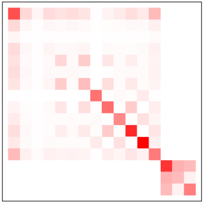


$d = 7, \; N_d = 70 + 66$

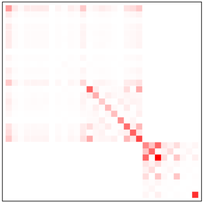


$d = 8, \; N_d = 95 + 102$

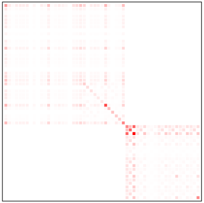
$\mathbf{Z} \in \mathbb{R}^{N_d \times N_d}$ matrix of size $N_d = \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$



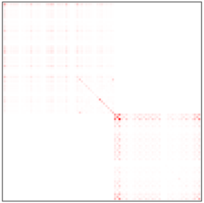
$d = 3,\ N_d = 13 + 3$



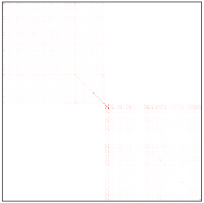
$d = 4,\ N_d = 22 + 9$



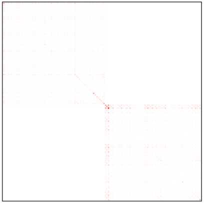
$d = 5,\ N_d = 34 + 21$



$d = 6,\ N_d = 50 + 39$



$d = 7,\ N_d = 70 + 66$



$d = 8,\ N_d = 95 + 102$

Flat extension criterion

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^{d} = f_{\mathrm{mom}}^{d} = f^{*}$?

Theorem [Hubert,M,Moustrou,Riener'22]

Let $\tilde{d} \le d \in \mathbb{N}$ and $\mathfrak{h}_{\mathcal{W}} := \max\{\langle e_i, \rho_0^{\vee}\rangle \,|\, 1 \le i \le n\}$.

Assume $\mathbf{Z}^{(\tilde{d})} \supseteq \mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})}$ and $\mathrm{rk}(\mathbf{Z}^{(\tilde{d})}) = \mathrm{rk}(\mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})})$.

Then $f_{\mathrm{mom}}^{d} = f^{*}$.

Additionally, if $\mathrm{tr}\big(\mathbf{X}^{(d)}\,\mathbf{Z}^{(d)}\big) = 0$, then $f_{\mathrm{sos}}^{d} = f^{*}$.

Proof: Adapt [Laurent,Mourrain'09] with different border basis. ■

Flat extension criterion

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^{d} = f_{\mathrm{mom}}^{d} = f^{*}$?

Theorem [Hubert,M,Moustrou,Riener'22]

Let $\tilde{d} \leq d \in \mathbb{N}$ and $\mathfrak{h}_{\mathcal{W}} := \max\{\langle e_i, \rho_0^{\vee}\rangle \mid 1 \leq i \leq n\}$.

Assume $\mathbf{Z}^{(\tilde{d})} \supseteq \mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})}$ and $\mathrm{rk}(\mathbf{Z}^{(\tilde{d})}) = \mathrm{rk}(\mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})})$.

Then $f_{\mathrm{mom}}^{d} = f^{*}$.

Additionally, if $\mathrm{tr}\left(\mathbf{X}^{(d)}\,\mathbf{Z}^{(d)}\right) = 0$, then $f_{\mathrm{sos}}^{d} = f^{*}$.

Proof: Adapt [Laurent,Mourrain'09] with different border basis. ■

Flat extension criterion

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^d = f_{\mathrm{mom}}^d = f^*$?

Theorem [Hubert,M,Moustrou,Riener'22]

Let $\tilde{d} \leq d \in \mathbb{N}$ and $\mathfrak{h}_{\mathcal{W}} := \max\{\langle e_i, \rho_0^\vee \rangle \mid 1 \leq i \leq n\}$.

Assume $\mathbf{Z}^{(\tilde{d})} \supseteq \mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})}$ and $\mathrm{rk}(\mathbf{Z}^{(\tilde{d})}) = \mathrm{rk}(\mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})})$.

Then $f_{\mathrm{mom}}^d = f^*$.

Additionally, if $\mathrm{tr}\left(\mathbf{X}^{(d)}\, \mathbf{Z}^{(d)}\right) = 0$, then $f_{\mathrm{sos}}^d = f^*$.

Proof: Adapt [Laurent,Mourrain'09] with different border basis. ∎

Flat extension criterion

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^{d} = f_{\mathrm{mom}}^{d} = f^*$?

Theorem [Hubert,M,Moustrou,Riener'22]

Let $\tilde{d} \leq d \in \mathbb{N}$ and $\mathfrak{h}_{\mathcal{W}} := \max\{\langle e_i, \rho_0^\vee \rangle \mid 1 \leq i \leq n\}$.

Assume $\mathbf{Z}^{(\tilde{d})} \supseteq \mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})}$ and $\mathrm{rk}(\mathbf{Z}^{(\tilde{d})}) = \mathrm{rk}(\mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})})$.

Then $f_{\mathrm{mom}}^{d} = f^*$.

Additionally, if $\mathrm{tr}\left(\mathbf{X}^{(d)}\, \mathbf{Z}^{(d)}\right) = 0$, then $f_{\mathrm{sos}}^{d} = f^*$.

Proof: Adapt [Laurent,Mourrain'09] with different border basis. ∎

Flat extension criterion

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^d = f_{\mathrm{mom}}^d = f^*$?

Theorem [Hubert,M,Moustrou,Riener'22]

Let $\tilde{d} \leq d \in \mathbb{N}$ and $\mathfrak{h}_{\mathcal{W}} := \max\{\langle e_i, \rho_0^\vee \rangle \mid 1 \leq i \leq n\}$.

Assume $\mathbf{Z}^{(\tilde{d})} \supseteq \mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})}$ and $\mathrm{rk}(\mathbf{Z}^{(\tilde{d})}) = \mathrm{rk}(\mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})})$.

Then $f_{\mathrm{mom}}^d = f^*$.

Additionally, if $\mathrm{tr}\big(\mathbf{X}^{(d)}\,\mathbf{Z}^{(d)}\big) = 0$, then $f_{\mathrm{sos}}^d = f^*$.

Proof: Adapt [Laurent,Mourrain'09] with different border basis. ∎

Flat extension criterion

Summary: There is an algorithm with input f and output $\mathbf{X}, \mathbf{y}, \mathbf{Z}$.

Under what conditions on $\mathbf{X}, \mathbf{y}, \mathbf{Z}$ is $f_{\mathrm{sos}}^{d} = f_{\mathrm{mom}}^{d} = f^*$?

Theorem [Hubert,M,Moustrou,Riener'22]

Let $\tilde{d} \leq d \in \mathbb{N}$ and $\mathfrak{h}_{\mathcal{W}} := \max\{\langle e_i, \rho_0^\vee \rangle \mid 1 \leq i \leq n\}$.

Assume $\mathbf{Z}^{(\tilde{d})} \supseteq \mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})}$ and $\mathrm{rk}(\mathbf{Z}^{(\tilde{d})}) = \mathrm{rk}(\mathbf{Z}^{(\tilde{d}-n+1-\mathfrak{h}_{\mathcal{W}})})$.

Then $f_{\mathrm{mom}}^{d} = f^*$.

Additionally, if $\mathrm{tr}\left(\mathbf{X}^{(d)}\,\mathbf{Z}^{(d)}\right) = 0$, then $f_{\mathrm{sos}}^{d} = f^*$.

Proof: Adapt [Laurent,Mourrain'09] with different border basis. ■

Thank You.