Symmetry in Trigonometric Optimization

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Introductory Example

The goal of trigonometric optimization is to find the global minimum of a function $\mathbb{R}^n \to \mathbb{R}$ such as

$$
\begin{aligned}
&-1 + 2/3\,(2\,\cos(2\pi x)\,\cos((-2x-2y)\pi)^2\,\cos(2\pi y) + 2\,\cos(2\pi x)\,\cos(2\pi y)^2\,\cos((-2x-2y)\pi)\\
&+2\,\cos(2\pi x)^2\,\cos(2\pi y)\,\cos((-2x-2y)\pi) + \cos(2\pi y)^2\,\cos((-2x-2y)\pi)^2 + \sin(2\pi x)^2\\
&\sin((-2x-2y)\pi)^2 + \cos(2\pi x)^2\,\cos((-2x-2y)\pi)^2 + \sin(2\pi x)^2\,\sin(2\pi y)^2 + \cos(2\pi x)^2\,\cos(2\pi y)^2\\
&-\sin(2\pi y)\,\sin((-2x-2y)\pi) - \cos(2\pi y)\,\cos((-2x-2y)\pi) - \sin(2\pi x)\,\sin((-2x-2y)\pi)\\
&-\cos(2\pi x)\,\cos((-2x-2y)\pi) - \sin(2\pi x)\,\sin(2\pi y) - \cos(2\pi x)\,\cos(2\pi y) + \sin(2\pi y)^2\\
&\sin((-2x-2y)\pi)^2 + 2\,\cos(2\pi x)\,\cos(2\pi y)\,\sin(2\pi x)\,\sin((-2x-2y)\pi) + 2\,\cos(2\pi x)\\
&\cos(2\pi y)\,\sin(2\pi x)\,\sin(2\pi y) + 2\,\cos(2\pi y)\,\cos((-2x-2y)\pi)\,\sin(2\pi y)\,\sin((-2x-2y)\pi)\\
&+2\,\sin(2\pi x)\,\sin((-2x-2y)\pi)\,\cos(2\pi y)\,\cos((-2x-2y)\pi) + 2\,\cos(2\pi x)\,\cos((-2x-2y)\pi)\\
&\sin(2\pi y)\,\sin((-2x-2y)\pi) + 2\,\cos(2\pi x)\,\cos((-2x-2y)\pi)\,\sin(2\pi x)\,\sin((-2x-2y)\pi)\\
&+2\,\sin(2\pi x)\,\sin(2\pi y)\,\cos(2\pi y)\,\cos((-2x-2y)\pi) + 2\,\sin(2\pi x)\,\sin(2\pi y)\,\cos(2\pi x)\\
&\cos((-2x-2y)\pi) + 2\,\cos(2\pi x)\,\cos(2\pi y)\,\sin(2\pi y)\,\sin((-2x-2y)\pi) + 2\,\sin(2\pi x)^2\,\sin(2\pi y)\\
&\sin((-2x-2y)\pi) + 2\,\sin(2\pi x)\,\sin(2\pi y)^2\,\sin((-2x-2y)\pi) + 2\,\sin(2\pi x)\,\sin((-2x-2y)\pi)^2\,\sin(2\pi y)).
\end{aligned}
$$

By exploiting *algebraic structures*, one can *simplify* the problem:

Here, we can rewrite the function as a polynomial $\boxed{6z^2 - 2z - 1}$!

Content

1. From trigonometric to generalized Chebyshev polynomials
2. The image of the generalized cosines as a semi–algebraic set
3. Optimization with Chebyshev polynomials in practice

The presented results are based on joint work with
Evelyne Hubert (Centre Inria d'Université Côte d'Azur),
Philippe Moustrou (Université Toulouse Jean Jaures),
Cordian Riener (UiT The Arctic University).

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From trigonometric to generalized Chebyshev polynomials

Trigonometric optimization

Let $\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n \leqslant \mathbb{R}^n$ be a lattice and $\langle \cdot, \cdot \rangle$ be the Euclidean scalar product.

$$\mathfrak{e}^{\mu}\,\mathfrak{e}^{\nu} = \mathfrak{e}^{\mu+\nu}$$
$$\mathfrak{e}^{\mu}\,\mathfrak{e}^{-\mu} = \mathfrak{e}^{0}$$

The algebra of trigonometric polynomials

For $\mu \in \Omega$, define $\mathfrak{e}^{\mu} : \mathbb{R}^n \to \mathbb{C}$ with

$$\mathfrak{e}^{\mu}(u) := \exp(-2\pi \mathrm{i}\,\langle \mu, u \rangle)$$

and write $\mathbb{R}[\Omega] = \mathbb{R}[\mathfrak{e}^{\pm\omega_1}, \ldots, \mathfrak{e}^{\pm\omega_n}]$.

$$f = \sum_{\mu} f_{\mu}\,\mathfrak{e}^{\mu} \in \mathbb{R}[\Omega]$$

$$\mu = \textstyle\sum_i \alpha_i\,\omega_i \in \Omega$$
$$\Rightarrow \mathfrak{e}^{\mu} = \textstyle\prod_i (\mathfrak{e}^{\omega_i})^{\alpha_i}$$

Periodicity

Let $\Lambda := \{\lambda \in \mathbb{R}^n \,|\, \forall\, \mu \in \Omega : \langle \mu, \lambda \rangle \in \mathbb{Z}\}$ be the **dual lattice**.
Then, for $f \in \mathbb{R}[\Omega]$, $\lambda \in \Lambda$, $u \in \mathbb{R}^n$, we have $f(u + \lambda) = f(u)$.

The trigonometric optimization problem

For $f = \sum_{\mu} f_{\mu}\,\mathfrak{e}^{\mu} \in \mathbb{R}[\Omega]$ with $f_{\mu} = f_{-\mu} \in \mathbb{R}$, find $f^* := \min_{u \in \mathbb{R}^n} f(u)$.

Symmetry in trigonometric optimization

Let $\mathcal{W} \leqslant \mathrm{O}_n(\mathbb{R})$ be a finite orthogonal group and
Ω be a **$\mathcal{W}$–lattice**, that is, for $A \in \mathcal{W}$, $\mu \in \Omega$, we have $A\,\mu \in \Omega$.

The linear action of $\mathcal{W}$ on $\mathbb{R}[\Omega]$

$$\begin{array}{rccc} \cdot : & \mathcal{W} \times \mathbb{R}[\Omega] & \to & \mathbb{R}[\Omega], \\ & (A, \mathfrak{e}^{\mu}) & \mapsto & \mathfrak{e}^{A\,\mu} \end{array}$$

- Say f is **$\mathcal{W}$–invariant**, if $\mathcal{W} \cdot f = \{f\}$
- $\mathbb{R}[\Omega]^{\mathcal{W}}$ the **algebra of $\mathcal{W}$–invariants**

$$A \cdot \sum_{\mu} f_{\mu}\, \mathfrak{e}^{\mu} = \sum_{\mu} f_{\mu}\, \mathfrak{e}^{A\,\mu}$$

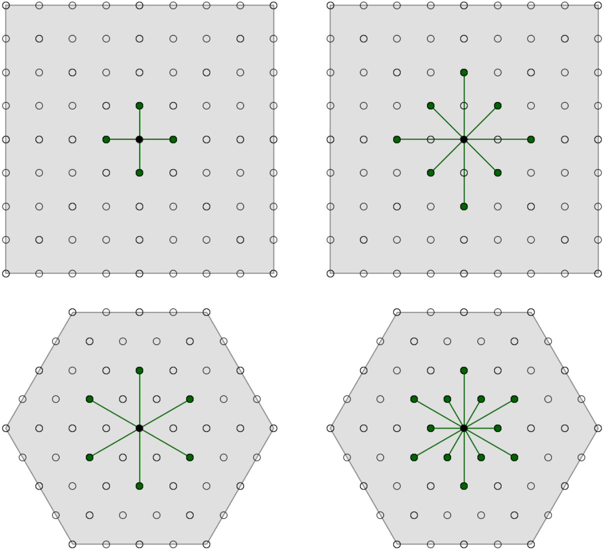
$$A \cdot (f\, g) = (A \cdot f)(A \cdot g)$$

$$A \cdot (f + g) = A \cdot f + A \cdot g$$

Generators (Lorenz'05: Multiplicative Invariant Theory)

- As a space, $\mathbb{R}[\Omega]^{\mathcal{W}}$ is generated by the $\frac{1}{|\mathcal{W}|} \sum\limits_{A \in \mathcal{W}} \mathfrak{e}^{A\,\mu}$, $\mu \in \Omega$.
- As an algebra, $\mathbb{R}[\Omega]^{\mathcal{W}}$ is finitely generated.

Root systems, Weyl groups and lattices (Example)



Such lattices Ω and groups $\mathcal{W}$ arise from, e.g., root systems $\subseteq \mathbb{R}^n$.

Root systems, Weyl groups and lattices (Definition)

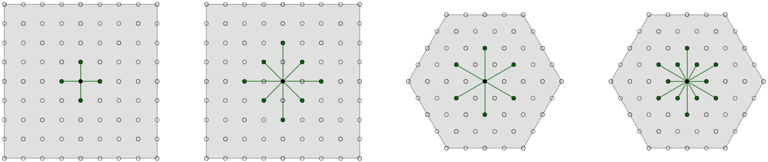
$\mathrm{R} \subseteq \mathbb{R}^n$ root system (Bourbaki'68 Ch. VI: Systèmes de Racines)

R1 R is finite, spans $\mathbb{R}^n$ and does not contain 0.

R2 If $\rho, \tilde{\rho} \in \mathrm{R}$, then $\langle \tilde{\rho}, \rho^\vee \rangle \in \mathbb{Z}$, where $\rho^\vee := 2\,\rho / \langle \rho, \rho \rangle$.

R3 If $\rho, \tilde{\rho} \in \mathrm{R}$, then $A_\rho(\tilde{\rho}) \in \mathrm{R}$, where $A_\rho(u) := u - \langle u, \rho^\vee \rangle \rho$.

- The **Weyl group** $\mathcal{W}$ is the group generated by the A_ρ.
- The **coroot lattice** Λ is the lattice spanned by the $\rho^\vee$.
- The **weight lattice** Ω is the dual lattice of Λ.



What are the generators of $\mathbb{R}[\Omega]^{\mathcal{W}}$ (as an algebra)?

Generalized Chebyshev polynomials

The generalized cosine functions

For $\mu \in \Omega$, define $\mathfrak{c}_\mu \in \mathbb{R}[\Omega]^{\mathcal{W}}$ with

$$\mathfrak{c}_\mu(u) := \frac{1}{|\mathcal{W}|} \sum_{A \in \mathcal{W}} \mathfrak{e}^{A\mu}(u).$$

$$\Omega = \mathbb{Z}\,\omega_1 \oplus \ldots \oplus \mathbb{Z}\,\omega_n$$

$$\mathbb{R}[\Omega] = \mathbb{R}[\mathfrak{e}^{\pm\omega_1}, \ldots, \mathfrak{e}^{\pm\omega_n}]$$

The algebra of $\mathcal{W}$–invariants (Bourbaki'68 Ch. VI)

- The $\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}$ are algebraically independent.
- $\mathbb{R}[\Omega]^{\mathcal{W}} = \mathbb{R}[\mathfrak{c}_{\omega_1}, \ldots, \mathfrak{c}_{\omega_n}]$ is a polynomial algebra.

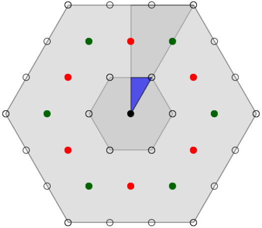
The generalized Chebyshev polynomial associated to $\mu \in \Omega$

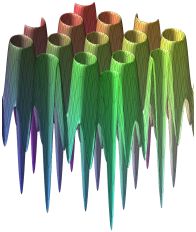
$T_\mu \in \mathbb{R}[z] = \mathbb{R}[z_1, \ldots, z_n]$, so that $T_\mu(\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) = \mathfrak{c}_\mu(u)$.

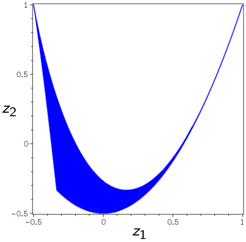
Example ($n = 1$, $\Omega = \mathbb{Z}$)

$\mathbb{R}[\mathfrak{e}^{\pm 1}(u)]^{\{\pm 1\}} = \mathbb{R}[\cos(2\pi u)]$ and $T_\mu(\cos(2\pi u)) = \cos(2\pi\mu u)$.

Rewriting the trigonometric optimization problem







Example (Ω hexagonal lattice, $\mathcal{W} = \mathfrak{D}_6$ dihedral group)

For $S := \mathcal{W}\{2\omega_1, \omega_2\}$ and $f_{2\omega_1} := 1$, $f_{\omega_2} := 2$, we have

$$\min_{u\in\mathbb{R}^2} \sum_{\mu\in S} f_\mu \, \mathfrak{c}_\mu(u) = \min_{z\in\mathcal{T}} T_{2\omega_1}(z) + 2\, T_{\omega_2}(z) = \min_{z\in\mathcal{T}} 6\, z_1^2 - 2\, z_1 - 1 = -\frac{7}{6}$$

New feasible region: The image of the generalized cosines

$$\mathcal{T} := \{\mathfrak{c}(u) := (\mathfrak{c}_{\omega_1}(u), \ldots, \mathfrak{c}_{\omega_n}(u)) \mid u \in \mathbb{R}^n\}$$

The image of the generalized cosines as a semi–algebraic set

$\rightarrow$ (Hubert, M, Riener'22)

Appearances of $\mathcal{T}$ in the literature

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The transformation

(3.10) $u=x+y,\ v=xy$

maps the region R in the (x, y)-plane in a one-to-one and regular way onto the region $\{(u,v) \mid -v-1<u<v+1,\ u^2-4v>0\}$, which will also be denoted by R (cf. fig. 4).

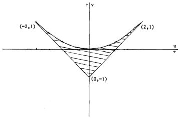

Fig. 4.

This region is bounded by two perpendicular lines and a parabola which touches the two lines. Let the weight function $\mu^{\alpha,\beta,\gamma}$ be defined by

Koornwinder'74

In [13], the existence of the Gaussian cubature rule in the sense of m-degree and the connection to orthogonal polynomials were established in the context of compact simple Lie groups. The case of the group G_2 was used as an example, where a numerical example was given. The domain Δ^* and the one in [13] differ by an affine change of variables.

Our results give explicit nodes and weights of the cubature rule and provide further explanation for the result.

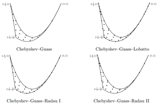

Figure 6.1. The cubature nodes on the region Δ^*.

6.2 Gauss–Lobatto cubature and Chebyshev polynomials of the first kind

In the case of $w_{-\frac{1}{2},-\frac{1}{2}}$, the change of variables $\mathbf{t} \mapsto x$ shows that (3.22) leads to a cubature of m-degree $2n-1$ based on the nodes of Y_n.

Xu'10

J Fourier Anal Appl (2010) 16: 383–433 427

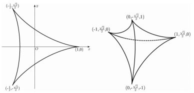

Fig. 10 The region Δ^* for $d=2$ and $d=3$

We will need the cases of $\alpha=-1/2$ and $\alpha=1/2$ of the weighted inner product

$$\langle f, g\rangle_{w^\alpha} := c_\alpha \int_{\Delta^*} f(z)\overline{g(z)}w^\alpha(z)dx,$$

where c_α is a normalization constant, $c_\alpha := 1/\int_{\Delta^*} w^\alpha(z)dx$. The change of variables $\mathbf{t} \mapsto x$ shows immediately that

Xu'12

Through the Kaleidoscope 35

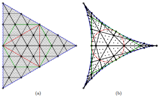

Figure 1.5 The equilateral domain Δ in (a) maps to the Deltoid δ in (b) under $t \mapsto z(t)$.

Continuous orthogonality. Let Φ be an irreducible root system on $\mathcal{V} = \mathbb{R}^d$ with an alcove $\triangle$ being the simplex defined in Lemma 1.21. The corresponding family of multivariate Chebyshev polynomials are orthogonal on the domain

Munthe–Kaas'12

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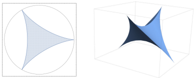


Fig. 1. The figure on the left corresponds to the orthogonality region for the case $n=2$. This is the area enclosed by Steiner's hypocycloid, which is given by an algebraic curve of fourth degree (8.2). The figure on the right is the three-dimensional region of orthogonality for $n=3$ which is determined by the algebraic equation of degree six (8.3).

$$\mathrm{vol}(\phi(A_c)) = \int_{\phi(A_c)} d\phi = \frac{(2\sqrt{\pi})^n}{\Gamma(1+\frac{n}{2})\prod_{k=1}^{n}\binom{n+1}{k}}.$$

For $n=2$ we obtain the area of Steiner's hypocycloid, which is $4\pi/9$. For $n=3$ we obtain the volume of the 3-dimensional analog of Steiner's hypocycloid, which equals $\pi/9$. See Fig. 1.

Koelink'20

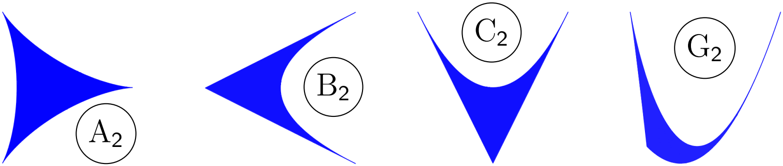
Semi–algebraic description

If R is of type A_{n-1}, B_n, C_n, D_n or G_2, then there exists a symmetric matrix polynomial $H \in \mathbb{R}[z]^{n\times n}$, such that

$$\mathcal{T} = \{z \in \mathbb{R}^n \,|\, H(z) \succeq 0\}.$$

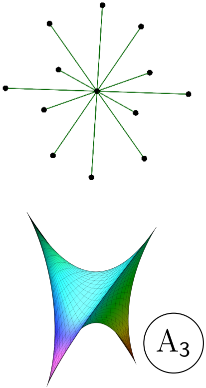
The closed formula in the Chebyshev basis is

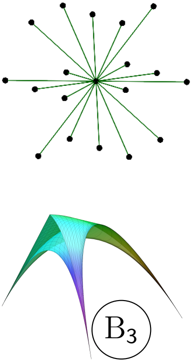
$$H = \begin{pmatrix} (T_0 - T_{2\omega_1})/2 & (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & \cdots \\ (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & \cdots \\ (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & (2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1})/32 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

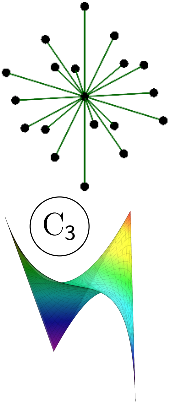


Describing $\mathcal{T}$ for irreducible root systems

$$H = \begin{pmatrix} (T_0 - T_{2\omega_1})/2 & (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & \cdots \\ (T_{\omega_1} - T_{3\omega_1})/4 & (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & \cdots \\ (T_0 - T_{4\omega_1})/8 & (2T_{\omega_1} - T_{3\omega_1} - T_{5\omega_1})/16 & (2T_0 + T_{2\omega_1} - 2T_{4\omega_1} - T_{6\omega_1})/32 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$







Optimization with Chebyshev polynomials in practice

→ (Hubert, M, Moustrou, Riener'22)

From trigonometric to polynomial optimization

Let $\mathcal{W}$ be a Weyl group, Ω the weight lattice and $f \in \mathbb{R}[\Omega]^{\mathcal{W}}$.

Rewriting to a polynomial optimization problem

We seek $f^* = \min_{z \in \mathcal{T}} \sum_{\mu} f_\mu \, T_\mu(z) = \min_{H(z) \succeq 0} \sum_{\mu} f_\mu \, T_\mu(z)$.

- (Lasserre'01) moment/sums of squares hierarchy, based on Putinar's Positivstellensatz'93
- (Henrion, Lasserre'06) ... with matrix inequalities, based on the Hol–Scherer Positivstellensatz'05

Matrix SOS reinforcement

$$\begin{aligned} f^* &= \min \quad \sum_\mu f_\mu \, T_\mu(z) \\ &\quad \text{s.t.} \quad z \in \mathbb{R}^n,\ H(z) \succeq 0 \\ &= \max \quad r \\ &\quad \text{s.t.} \quad r \in \mathbb{R},\ \forall\, H(z) \succeq 0 : \\ &\qquad\qquad \sum_\mu f_\mu \, T_\mu(z) - r \geq 0. \\ &\geq \sup \quad r \\ &\quad \text{s.t.} \quad r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathbb{R}[z]),\ Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n}), \\ &\qquad\qquad \sum_\mu f_\mu \, T_\mu - r = q + \mathrm{tr}(H\,Q) \end{aligned}$$

Write $Q \in \mathrm{SOS}(\mathbb{R}[z]^{n\times n})$, if

$\exists\, Q_1, \ldots, Q_k \in \mathbb{R}[z]^n$, s.t.

$$Q(z) = \sum_{i=1}^k Q_i(z)\, Q_i(z)^t$$

For computations, restrict q, Q to finite space ($d \in \mathbb{N}$)

$$\mathcal{F}_d := \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$$

$$T_\mu \, T_\nu = \sum_{\langle \omega, \rho_0^\vee \rangle \leq \langle \mu+\nu, \rho_0^\vee \rangle} t_\omega \, T_\omega$$

If $T_\mu \in \mathcal{F}_{d_1}$ and $T_\nu \in \mathcal{F}_{d_2}$,
then $T_\mu \, T_\nu \in \mathcal{F}_{d_1+d_2}$.

Semi–definite lower bounds

SOS hierarchy for trigonometric polynomials with $\mathcal{W}$–symmetry

For $d \in \mathbb{N}$ sufficiently large and $\mathcal{F}_d = \langle T_\mu \mid \langle \mu, \rho_0^\vee \rangle \leq d \rangle_\mathbb{R}$, we have

$$f^* \geq f_{\mathrm{sym}}^d := \begin{array}{rl} \sup & r \\ \text{s.t.} & r \in \mathbb{R},\ q \in \mathrm{SOS}(\mathcal{F}_d),\ Q \in \mathrm{SOS}(\mathcal{F}_{d-n}^{n\times n}), \\ & \sum\limits_{\mu} f_\mu\, T_\mu - r = q + \mathrm{tr}(H\, Q). \end{array}$$

Then $f_{\mathrm{sym}}^d \leq f_{\mathrm{sym}}^{d+1}$ and $\lim\limits_{d\to\infty} f_{\mathrm{sym}}^d = f^*$.

Translation to an SDP → Maple

Compute $A_0,\ A_\mu \in \mathrm{Sym}^{N(d)}$, such that

$$f_{\mathrm{sym}}^d = \begin{array}{rl} \sup & f_0 - \mathrm{tr}(A_0\, X) \\ \text{s.t.} & X \in \mathrm{Sym}_{\succeq 0}^{N(d)},\ \forall\, 0 \neq \mu : \\ & \mathrm{tr}(A_\mu\, X) = f_\mu. \end{array}$$

Matrix size:

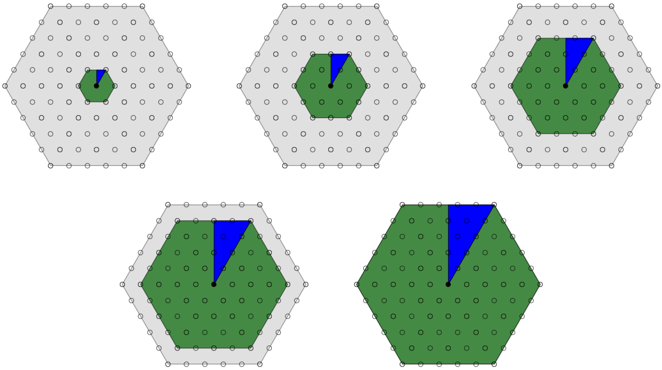
$$N(d) := \dim(\mathcal{F}_d) + n \dim(\mathcal{F}_{d-n})$$

Comparison with the dense approach

SOHS hierarchy for trigonometric polynomials without symmetry

For $f = \sum_\mu f_\mu \mathfrak{e}^\mu \in \mathbb{R}[\Omega]$ with $f_\mu = f_{-\mu} \in \mathbb{R}$, find $f^* := \min_{u \in \mathbb{R}^n} f(u)$.

(Dumitrescu'07) $f_{\text{dense}}^d := \sup\{r \in \mathbb{R} \mid f - r \in \mathrm{SOHS}(d)\} \to$ SDP.



Conclusion

Conclusion

Summary

1. The algebra of invariant trigonometric polynomials is again polynomial.
2. The objective function is rewritten in terms of generalized Chebyshev polynomials.
3. We optimize on the image of the generalized cosines, a semi–algebraic set.
4. We adapt Lasserre's hierachy in the Chebyshev basis with matrix constraints.

Work in progress

1. What is better from a qualitative point, the dense or symmetric approach?
2. How does it compare with symmetry adapted bases? (ISSAC 2023)
3. What is the convergence rate? (exponential vs polynomial?)

Thanks for your attention.

E. Hubert, T. Metzlaff, C. Riener: *Orbit spaces of Weyl groups acting on compact tori: a unified and explicit polynomial description*
https://hal.archives-ouvertes.fr/hal-03590007

E. Hubert, T. Metzlaff, P. Moustrou, C. Riener: *Optimization of trigonometric polynomials with crystallographic symmetry and spectral bounds for set avoiding graphs*
https://hal.archives-ouvertes.fr/hal-03768067

T. Metzlaff: *Symmetry adapted bases for trigonometric optimization*
to appear

T. Metzlaff: *Maple2022:GeneralizedChebyshev*
https://github.com/TobiasMetzlaff/GeneralizedChebyshev